A Statistical Fields Theory underlying the Thermodynamics of Ricci Flow and Gravity

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The paper proposes a statistical fields theory of quantum reference frame underlying the Perelman's analogies between his formalism of the Ricci flow and the thermodynamics. The theory is based on a $d = 4-\epsilon$ quantum non-linear sigma model (NLSM), interpreted as a quantum reference frame system which a to-be-studied quantum system is relative to. The statistic physics and thermodynamics of the quantum frame fields is studied by the density matrix obtained by the Gaussian approximation quantization. The induced Ricci flow of the frame fields and the Ricci-DeTurck flow of the frame fields associated with the density matrix is deduced. In this framework, the diffeomorphism anomaly of the theory has a deep thermodynamic interpretation. The trace anomaly is related to a Shannon entropy in terms of the density matrix, which monotonically flows and achieves its maximal value at the flow limit, called the Gradient Shrinking Ricci Soliton (GSRS), corresponding to a thermal equilibrium state of spacetime. A relative Shannon entropy w.r.t. the maximal entropy gives a statistical interpretation to Perelman's partition function, which is also monotonic and gives an analogous H-theorem to the statistical frame fields system. A temporal static 3-space of a GSRS 4-spacetime is also a GSRS in lower 3-dimensional, we find that it is in a thermal equilibrium state, and Perelman's analogies between his formalism and the thermodynamics of the frame fields in equilibrium can be explicitly given in the framework. Extending the validity of the Equivalence Principle to the quantum level, the quantum reference frame fields theory at low energy gives an effective theory of gravity, a scale dependent Einstein-Hilbert action plus a cosmological constant is recovered. As a possible underlying microscopic theory of the gravitational system, the theory is also applied to understand the thermodynamics of the Schwarzschild black hole.

I. INTRODUCTION

Recent works [1, 2] show possible relations between Perelman's formalism of the Ricci flow and some fundamental problems in quantum spacetime and quantum gravity, for instance, the trace anomaly and the cosmological constant problem. Perelman's seminal works (the section-5 of [3]) and further development by Li [4, 5] also suggest deep relations between the Ricci flow and the thermodynamics system, not only the irreversible non-equilibrium but also the thermal equilibrium thermodynamics of certain underlying microscopic system. In [3] Perelman also declared a partition function and his functionals without specifying what the underlying microscopic ensemble really are (in physics). So far it is not clear whether the beautiful thermodynamic analogies are physical or pure coincidences. On the other hand, inspired by the surprising analogies between the black hole and thermodynamics system, it is generally believed the existence of temperature and entropy of a black hole. Works along this line also showed, in many aspects, the gravitational system would be profoundly related to thermodynamics system (see recent review [6] and references therein), it is generally conjectured that there would exist certain underlying statistical theory for the underlying microscopic quantum degrees of freedom of gravity. It gradually becomes one of the touchstones for a quantum gravity.

The motivations of the paper are, firstly, to propose an underlying statistical fields theory for Perelman's seminal thermodynamics analogies of his formalism of the Ricci flow, and secondly, for understanding the possible microscopic origin of the spacetime thermodynamics especially for the Schwarzschild black hole. We hope the paper could push forward the understanding to the possible interplay of the mysterious Perelman's formalism of Ricci flow and the quantum spacetime and gravity. To our knowledge, several tentative works have been devoted to the goal, see e.g. [7–10], but frankly speaking, the physical picture underlying the Ricci flow is not fully clear, if a fundamental physical theory underlying the Ricci flow and a fundamental theory of quantum spacetime is lacking.

Based on our previous works [1, 2, 11–16] on the quantum reference frame and its relation to Perelman's formalism of the Ricci flow, we propose a statistical fields theory of the quantum reference frame as a possible underlying theory of Perelman's seminal analogies between his geometric functionals and the thermodynamic functions. In section II, we review the theory of quantum reference frame based on a $d = 4-\epsilon$ quantum non-linear sigma model, at the Gaussian approximation quantization, we obtain a density matrix of the frame fields system as a physical foundation to the

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statistical interpretation of the theory. The induced Ricci flow of the frame fields and the Ricci-DeTurck flow of the frame fields associated with the density matrix is deduced. In section III, we discuss the diffeomorphism and related trace anomaly of the quantum frame fields theory and its profound implications to the irreversible non-equilibrium thermodynamics of the frame fields, for instance, the statistical entropy and an analogous H-theorem of the frame fields, and the effective gravity theory at cosmic scale (especially the emergence of the cosmological constant). In section IV, the thermal equilibrium state of the frame fields as a flow limit configuration (the Gradient Shrinking Ricci Soliton) is discussed, in which the density matrix recovers the thermal equilibrium canonical ensemble density. This section gives a physical foundation to Perelman's seminal thermodynamic analogies. In section V, the framework gives a possible microscopic understanding of the thermodynamics of the Schwarzschild black hole. Finally, we summarize the paper and give conclusions in the section VI.

II. QUANTUM REFERENCE FRAME

Reference frame is one of the most fundamental notions in physics. Any measurement in physics is performed or described, a reference frame has always been explicitly or implicitly used. In classical physics, the reference frame is idealizationally used via classical rulers and clocks to label the spacetime coordinates, which are classical, external, and rigid without any fluctuation. Even in the textbook quantum mechanics or quantum fields theory, the spacetime coordinates are still classical. But quantum principles tell us that all physical measuring instruments including the rulers and clocks are inescapably quantum fluctuating. Such idealizational and classical treatment of reference frame works not bad in quantum mechanics and quantum fields theory. To a large extent, this is due to the fact that the general coordinates transformation and gravitational effects are not seriously taken into account. Just as expected, when the quantum principles are seriously applied to the spacetime itself and gravitational phenomenon, severe difficulties arise, e.g. information losses (non-unitary), diffeomorphism anomaly and the cosmological constant problems, etc.

The quantum reference frame is a recurring theme in literature (not completely list, see e.g. [17–24] and references therein) based on various difference physical motivations, from quantum foundation to quantum information or quantum communication, to quantum gravity. For example, in Ref.[17], the author suggests the general relation between superselection rules and the lack of reference frame. In Ref.[20], it also more practically shows that extra assumptions about the superselection rules can not be avoided from the viewpoint of quantum information and quantum communication theory, if local observers do not share common information about their relative phase or Cartesian frames etc. The extra assumptions of the superselection rules may be also viewed as the weakness of textbook quantum mechanics, which can be overcome by introducing appropriate quantum reference frame. And many models (e.g. [18, 22, 23]) of quantum reference frame and relational descriptions to the quantum system and the quantum reference frame as a whole are suggest into the quantum foundation. In recent works [24] and the references therein, the authors review three approaches (relational Dirac observables, the Page-Wootters formalism and quantum deparameterizations) of relational quantum dynamics and suggest their equivalence. Other author focus on the possible role of quantum reference frame to the decoherence in quantum gravity [21, 25]. Certainly, the works list of the direction is far from complete, which is beyond the scope and ability of the author.

Fundamentally, our work shares the similar philosophical viewpoint to the role of quantum reference frame in quantum mechanics, such as considering that an appropriate materialized (but idealized) reference frame obeying the same law of quantum mechanics must be taken into account, and in the full quantum theory a relational description based on an entanglement of a quantum system and the quantum reference frame as a whole must play a fundamental role. However, there are some differences from the past literature that we considered more important. First, we do not simply or merely treat the quantum clock as a quantum mechanical system ([23, 24]) (which is more simple and has less degree of free to deal with as discussed in most quantum reference literature, in fact our early work ([11, 12]) also started from the operationally treatment of quantum clock to get some general conclusions on the vaccum energy and the cosmological constant problem), but in the paper we put both quantum space-rod and clock-time on an equal footing in the framework of quantum statistical fields, so that it makes the theory more appropriate to incorporate gravity, under the assumption of a quantum version of equivalence principle. To my understanding, quantum clock can be viewed as a first step model and far from a theory. Second, based on the quantum spacetime reference frame model (i.e. the $d = 4 - \epsilon$ non-linear sigma model), our paper does not treat the genuine relational quantities from the very beginning (as most literature tend to announce), but we prepare the quantum frame fields of reference in a laboratory frame (the $d = 4 - \epsilon$ base spacetime of the non-linear sigma model) as the starting reference, and then quantum events are relative to the prepared quantum frame fields. In this sense, the framework equivalently assumes the existence of an external, classical and rigid (free from quantum fluctuation and volume fixed) reference frame to be the laboratory frame, since the non-linear sigma model allows us to assign quantum state of spacetime reference (the target spacetime) to the base spacetime to arbitrary precision. But it can be easily verified that the theory is

independent to the laboratory frame (metric, sign etc.) in the non-linear sigma model. The notion of the external and classical laboratory frame is just for convenience, since a quantum statistical fields theory is historically (or maybe more appropriate to) defined on an inertial frame (flat spacetime). So the relational quantities describing the relation between the quantum system and the quantum spacetime reference system is in essential in the framework. Third, also for the reason of the base spacetime independence of the non-linear sigma model, whose Hamiltonian is trivial, so the theory of spacetime reference frame is more properly quantized by using the path integral or functional method rather than the operator methods (e.g. the relational Dirac observables quantization or relational Schrodinger picture in Page-Wootters formalism). And fourth, there is a fundamentally non-unitary relation between two spacetime reference frames under a coordinate transformation due to an irreversible Ricci flow of spacetime reference frame, unlike most approaches in which the coordinate transformation between difference reference frames is assumed unitary. This is considered as a key ingredient of quantum spacetime reference frame that is intrinsically ensemble statistical and thermal.

Generally speaking, our approach alongs the general philosophy of the quantum reference frame but is considered independent to the details of the past literature. The framework associates to several elegant physics and mathematical structures that are not discussed in the past literature, such as the non-linear sigma model, Shannon entropy, the Ricci flow and density Riemannian geometry, etc. And our previous works [1, 2, 11–16] have revealed very rich consequencies of the framework, (e.g. the acceleration expansion of the late epoch universe, the cosmological constant, diffemorphism anomaly, the inflationary early universe, local conformal stability and non-collapsibility, modified gravity, etc.), but frankly speaking, the possible consequencies of the quantum reference frame are still far from fully discovered. The main motivation here for a quantum treatment of a reference frame system is that it might form a foundation to constructing a theory of quantum spacetime and quantum gravity that is analogous to the way it is used to construct the classical general relativity and it is crucial in understanding the microscopic origin of the spacetime thermodynamics.

A. Definition

In this section, we propose a quantum fields theory of reference frame as a starting point to study a quantum theory of spacetime and quantum gravity, based on an Equivalence Principle extended to reference frame described by quantum state (discussed by a paradox in Section-V-B and in the conclusion of the paper). The generalization of the Equivalence Principle to the quantum level might form another foundation to a quantum reference frame and a quantum gravity. How the Equivalence Principle behaves at the quantum level has many discussions having a long history (e.g. [26–30] and references therein, and [31, 32] for an extended thermal version). The Equivalence Principle is the physical foundation of measuring the spacetime by physical material reference frame even at the quantum level, and it is the bridge between the geometric curved spacetime and gravity, and hence the gravity is simply a relational phenomenon that the motion of a test particle in gravity is manifested as a relative motion w.r.t. the (quantum) material reference frame. Without the Equivalence Principle, we would lost the physical foundation of all these concepts. Therefore, the basic argument of the paper is that there are several supports (e.g. uniform quantum origin of the accelerating expansion of the universe posited by myself in previous works [1, 2, 13], and a consistent incorporating the thermodynamics of the spacetime shown in this work) and the self-consistency of the framework are all possible evidences for its validity for the quantum reference frame.

In this framework, a to-be-studied quantum system described by a state $|\psi\rangle$ and the spacetime reference system by $|X\rangle$ are both quantum. The states of the whole system are given by an entangled state

$$|\psi[X]\rangle = \sum_{ij} \alpha_{ij} |\psi\rangle_i \otimes |X\rangle_j \tag{1}$$

in their direct product Hilbert space $\mathcal{H}_\psi \otimes \mathcal{H}_X$. The state (1) of the to-be-studied system and the reference frame system is an entangled state but a trivial direct product state is for the reason of calibration between them. Usually, a quantum measurement is performed as follows. At a preparation step of a quantum measurement, a one-to-one correlation between a quantum system $|\psi\rangle_i$ and a reference system $|X\rangle_j$ (a quantum instrument or ruler) is prepared, called calibration. The step in usual sense is a comparison and adjustment of the measuring instrument $|X\rangle_j$ by a calibration standard $|\psi_{standard}\rangle_i$ which is physically similar with the to-be-studied system $|\psi\rangle_i \doteq |\psi_{standard}\rangle_i$. A well-calibrated entangled state $\sum_{ij} \alpha_{ij} |\psi_{standard}\rangle_i \otimes |X\rangle_j$ can be used to measure the to-be-studied system $|\psi\rangle_i$ with the reference to the quantum instrument $|X\rangle_j$. In essential, the measurement indirectly performs a comparision between $|\psi\rangle_i$ and the fiducial state $|\psi_{standard}\rangle_i$. So the entangled state $|\psi[X]\rangle$ is a superposition of all possible one-to-one correlations. According to the standard Copenhagen interpretation of a quantum state, the to-be-studied quantum system collapsing into a state $|\psi\rangle_i$ together with the collapsing of the quantum reference system into the corresponding

$|X\rangle_j$ happening by the joint probability $|\alpha_{ij}|^2$, meaning that when the state of the quantum instrument is read out being in state $|X\rangle_j$, then in this sense the to-be-studied system is inferred to be the corresponding $|\psi\rangle_i$. A simple and practical example is the Stern-Gerlach experiment (see [1]). The entangled state generalizes the textbook quantum description of the state $|\psi(x)\rangle$ w.r.t. an idealized parameter x of a classical reference system free from quantum fluctuations (in quantum mechanics x is the Newtonian time, in quantum fields theory x_a are the Minkowskian spacetime).

The entangled state $|\psi[X]\rangle$ is inseparable, so that the state can only be interpreted in a relational manner, i.e. the entangled state describes the "relation" between $|\psi\rangle$ and $|X\rangle$, but each absolute state. The individual state $|\psi\rangle$ has physical meaning only being reference to $|X\rangle$ entangled to it. When quantum mechanics is reformulated on the new foundation of the relational quantum state (the entangled state) describing the "relation" between the state of the under-studied quantum system and the state of the quantum reference system, a gravitational theory is automatically contained in the quantum framework without extra assumption.

Since the state of reference $|X\rangle$ is also subject to quantum fluctuation, so mathematically speaking, the state $|\psi[X]\rangle$ can be seen as the state $|\psi(x)\rangle$ with a smeared spacetime coordinates, instead of the textbook state $|\psi(x)\rangle$ with a definite and classical spacetime coordinates. The state $|\psi[X]\rangle$ could recover the textbook state $|\psi(x)\rangle$ only when the quantum fluctuation of the reference system is small enough and hence can be ignored. More precise, the 2nd order central moment (even higher order central moments) fluctuations of the spacetime coordinate $\langle \delta X^2\rangle$ (the variance) can be ignored compared with its 1st order moment of quadratic distance $\langle \Delta X\rangle^2$ (squared mean), where $\langle ...\rangle$ represents the quantum expectation value by the state of the reference system $|X\rangle$. In this 1st order approximation, this quantum framework recovers the standard textbook quantum mechanics without gravity. When the quantum fluctuation $\langle \delta X^2\rangle$ as the 2nd order correction of the reference frame system is important and taken into account, gravity as a next order effects emerges in the quantum framework, as if one introduces gravitation into the standard textbook quantum mechanics, details are seen below and in previous works.

To find the state $|X\rangle \in \mathcal{H}_X$ of the quantum reference system, a quantum theory of the reference frame must be introduced. If the quantum spacetime reference frame $|X^\mu\rangle$ ($\mu = 0, 1, 2, ...D-1$) itself is considered as the to-be-studied quantum system, w.r.t. the fiducial lab spacetime $|x_a\rangle$ as the reference system, ($a = 0, 1, 2, ...d-1$), the entangled state $|X(x)\rangle = \sum_{ij} \alpha_{ij} |X\rangle_i \otimes |x\rangle_j$ can be constructed by a mapping between the two states, i.e. $|x\rangle \to |X\rangle$. From the mathematical viewing point, to defined a D-dimensional manifolds we need to construct a non-linear differentiable mapping $X(x)$ from a local coordinate patch $x \in \mathbb{R}^d$ to a D-manifolds $X \in M^D$. The mapping in physics is usually realized by a kind of fields theory for $X(x)$, the non-linear sigma model (NLSM) [33–40]

$$S[X] = \frac{1}{2}\lambda \sum_{\mu,\nu=0}^{D-1} \int d^d x g_{\mu\nu} \sum_{a=0}^{d-1} \frac{\partial X^\mu}{\partial x_a}\frac{\partial X^\nu}{\partial x_a}, \tag{2}$$

where λ is a constant with dimension of energy density $[L^{-d}]$ taking the value of the critical density (68) of the universe.

In the action, x_a ($a = 0, 1, 2, ..., d-1$), with dimension length $[L]$, is called the base space in NLSM's terminology, representing the coordinates of the local patch. They will be interpreted as the lab wall and clock frame as the starting reference, which is considered fiducial and classical with infinite precision. For the reason that a quantum fields theory must be formulated in a classical inertial frame, i.e. flat Minkowskian or Euclidean spacetime, so the base space is considered flat. Without loss of generality, we consider the base space as the Euclidean one, i.e. $x \in \mathbb{R}^d$ which is better defined when one tries to quantize the theory.

The differential mapping $X_\mu(x)$, ($\mu = 0, 1, 2, ..., D-1$), with dimensional length $[L]$, is the coordinates of a general Riemannian or Lorentzian manifolds M^D (depending on the boundary condition) with curved metric $g_{\mu\nu}$, called the target space in NLSM's terminology. We will work with the real-defined coordinates for the target spacetime, and the Wick rotated version has been included into the general coordinates transformation of the time component. In the language of quantum fields theory, $X_\mu(x)$ or $X^\mu(x) = \sum_{\nu=0}^{D-1} g^{\mu\nu} X_\nu(x)$ are the real scalar frame fields.

Here, if not specifically mentioned, we will use the Einstein summation convention to sum over index variable appears twice (Latin index for the lab frame from 0 to $d-1$ and Greek index for the spacetime from 0 to $D-1$) and dropping the summation notation sigma.

From the physical point of view, the reference frame fields can be interpreted as a physical coordinates system by using particle/fields signals, for instance, a multi-wire proportional chamber that measuring coordinates of an event in a lab. To build a coordinates system, first we need to orient, align and order the array of the multi-wires with the reference to the wall of the lab x_a, ($a = 1, 2, 3$). The electron fields (ignoring the spin) in these array of multi-wires are considered as the scalar frame fields. With the reference to the wall of the lab, to locate a position of an event, at least three electron signals X_1, X_2, X_3 must be received and read in three orthogonal directions. The location information can be measured from the wave function of the electron fields, e.g. from the phase counting or particle number counting. Usually we could consider the electrons in the wires are free, and the field's intensity is not very

large, so that the intensity can be seen as a linear function of the coordinates of the lab's wall, $X_\mu(x) = \sum_{a=1}^3 e_\mu^a x_a$, $(\mu = 1,2,3)$, for instance, here $e_\mu^a = \delta_\mu^a$ is the intensity of the signals in each orthogonal direction. Meaning that when the direction μ is the lab's wall direction a, the intensity of the electron beam is 1, otherwise the intensity is 0. Similarly, one need to read an extra electron signal X_0 to know when the event happens, with the reference to the lab's clock x_0. Thus, the fields of these 3+1 electron signals can be given by

$$X_\mu(x) = \sum_{a=0}^{3} e_\mu^a x_a, \quad (\mu = 0,1,2,3). \tag{3}$$

The intensity of the fields e_μ^a is in fact the vierbein, describing a mapping from the lab coordinate x_a to the frame fields X_μ.

When the event happens at a long distance beyond the lab's scale, for instance, at the scale of earth or solar system, we could imagine that to extrapolate the multi-wire chamber to such long distance scale still seems OK, only replacing the electrons beam in wire by the light beam. However, if the scale is much larger than the solar system, for instance, to the galaxy or cosmic scale, when the signal travels along such long distance and be read by an observer, we could imagine that the broadening of the light beam fields or other particle fields gradually becomes non-negligible. More precisely, the 2nd (or higher) order central moment fluctuations of the frame fields signals can not be neglected, the distance of Riemannian/Lorentzian spacetime as a quadratic form must be modified by the 2nd moment fluctuation or variance $\langle \delta X^2 \rangle$ of the coordinates

$$\left\langle (\varDelta X)^2 \right\rangle = \langle \varDelta X \rangle^2 + \langle \delta X^2 \rangle. \tag{4}$$

A local distance element in spacetime is given by a local metric tensor at the point, so it is convenient to think of the location point X being fixed, and interpreting the variance of the coordinate affect only the metric tensor $g_{\mu\nu}$ at the location point. As a consequence, the expectation value of a metric tensor $g_{\mu\nu}$ is corrected by the 2nd central moment quantum fluctuation of the frame fields

$$\langle g_{\mu\nu} \rangle = \left\langle \frac{\partial X_\mu}{\partial x_a} \frac{\partial X_\nu}{\partial x_a} \right\rangle = \left\langle \frac{\partial X_\mu}{\partial x_a} \right\rangle \left\langle \frac{\partial X_\nu}{\partial x_a} \right\rangle + \frac{1}{2} \frac{\partial^2}{\partial x_a^2} \langle \delta X_\mu \delta X_\nu \rangle = g_{\mu\nu}^{(1)}(X) - \delta g_{\mu\nu}^{(2)}(X), \tag{5}$$

where

$$g_{\mu\nu}^{(1)}(X) = \left\langle \frac{\partial X_\mu}{\partial x_a} \right\rangle \left\langle \frac{\partial X_\nu}{\partial x_a} \right\rangle = \langle e_\mu^a \rangle \langle e_\nu^a \rangle \tag{6}$$

is the 1st order moment (mean value) contribution to the classical spacetime. For the contribution of the 2nd order central moment $\delta g_{\mu\nu}^{(2)}$ (variance), the expectation value of the metric generally tends to be curved up and deformed, the longer the distance scale the more important the broadening of the frame fields, making the spacetime geometry gradually deform and flow at long distance scale.

Since the classical solution of the frame fields (3) given by the vierbein satisfying the classical equation of motion of the NLSM, it is a frame fields interpretation of NLSM in a lab: the base space of NLSM is interpreted as a starting reference by the lab's wall and clock, the frame fields $X(x)$ on the lab are the physical instruments measuring the spacetime coordinates. In this interpretation we consider $d = 4 - \epsilon$, $(0 < \epsilon \ll 1)$ in (2) and $D = 4$ is the least number of the frame fields.

There are several reason why d is not precise but very close to 4 in the quantum frame fields interpretation of NLSM. d must be very close to 4, first, certainly at the scale of lab it is our common sense; Second if we consider the entangled system $\mathcal{H}_\psi \otimes \mathcal{H}_X$ between the to-be-studied physical system and the reference frame fields system, without loss of generality, we could take a scalar field ψ as the to-be-studied (matter) system, which shares the common base space with the frame fields, the total action of the two entangled system is a direct sum of each system

$$S[\psi, X] = \int d^d x \left[\frac{1}{2} \frac{\partial \psi}{\partial x_a} \frac{\partial \psi}{\partial x_a} - V(\psi) + \frac{1}{2} \lambda g_{\mu\nu} \frac{\partial X^\mu}{\partial x_a} \frac{\partial X^\nu}{\partial x_a} \right], \tag{7}$$

where $V(\psi)$ is some potential of the ψ fields. It can be interpreted as an action of a quantum fields ψ on general spacetime coordinates X. Since both ψ field and the frame fields X share the same base space x, here they are described w.r.t. the lab spacetime x as the textbook quantum fields theory defined on inertial frame x. If we interpret the frame fields as the physical general spacetime coordinates, the coordinates of ψ field must be transformed from inertial frame

x to general coordinates X. At the semi-classical level, or 1st moment approximation when the fluctuation of X can be ignored, it is simply a classical coordinates transformation

$$\begin{aligned} S[\psi, X] \overset{(1)}{\approx} S[\psi(X)] &= \int d^4X\sqrt{|\det g^{(1)}|}\left[\frac{1}{4}\left\langle g^{(1)}_{\mu\nu}\frac{\partial X^\mu}{\partial x_a}\frac{\partial X^\nu}{\partial x_a}\right\rangle\left(\frac{1}{2}g^{(1)\mu\nu}\frac{\delta\psi}{\delta X^\mu}\frac{\delta\psi}{\delta X^\nu}+2\lambda\right)-V(\psi)\right] \\ &= \int d^4X\sqrt{|\det g^{(1)}|}\left[\frac{1}{2}g^{(1)\mu\nu}\frac{\delta\psi}{\delta X^\mu}\frac{\delta\psi}{\delta X^\nu}-V(\psi)+2\lambda\right], \end{aligned} \tag{8}$$

in which $\overset{(1)}{\approx}$ stands for the 1st moment or semi-classical approximation, and $\frac{1}{4}\left\langle g^{(1)}_{\mu\nu}\frac{\partial X^\mu}{\partial x_a}\frac{\partial X^\nu}{\partial x_a}\right\rangle = \frac{1}{4}\left\langle g^{(1)}_{\mu\nu}g^{(1)\mu\nu}\right\rangle = \frac{1}{4}D = 1$ has been used. It is easy to see, at the semi-classical level, i.e. only consider the 1st moment of X while 2nd moment fluctuations are ignored, the (classical) coordinates transformation reproduces the scalar field action in general coordinates X up to a constant 2λ, and the derivative $\frac{\partial}{\partial x_a}$ is formally replaced by the functional derivative $\frac{\delta}{\delta X^\mu}$. $\sqrt{|\det g^{(1)}|}$ is the Jacobian determinant of the coordinate transformation, note that the determinant requires the coordinates transformation matrix to be a square matrix, so at semi-classical level d must be very close to $D = 4$, which is not necessarily true beyond the semi-classical level, when the 2nd moment quantum fluctuations are important. For instance, since d is a parameter but an observable in the theory, it could even not necessary be an integer but effectively fractal at the quantum level.

d not precisely 4 is for the quantum and topological reasons. To investigate this, we note that quantization depends on the homotopy group $\pi_d(M^D)$ of the mapping $X(x) : \mathbb{R}^d \to M^D$. If we consider the (Wick rotated) spacetime M^D topologically the S^D for simplicity, the homotopy group is trivial for all $d < D = 4$, in other words, when $d < 4$ the mapping $X(x)$ will be free from any unphysical singularities for topological reason, in this situation, the target spacetime is always mathematically well-defined. However, the situation $d = 4$ is a little subtle, since $\pi_4(S^4) = \mathbb{Z}$ is non-trivial, the mapping might meet intrinsic topological obstacle and become singular, i.e. a singular spacetime configuration. When the quantum principle is taken into account, this situation can not be avoided, and by its RG flow the spacetime is possibly deformed into intrinsic singularities making the theory ill-defined at the quantum level and non-renormalizable (RG flow not converge). So at the quantum level, $d = 4$ should be not precisely, we have to assume $d = 4 - \epsilon$ when the quantum principle applies, while at the classical or semi-classical level, considering $d = 4$ has no serious problem. The above argument is different from the conventional simple power counting argument, which claims the NLSM is perturbative non-renormalizable when $d > 2$, but it is not necessarily the case, it is known that numerical calculations also support $d = 3$ and $d = 4 - \epsilon$ are non-perturbative renormalizable and well-defined at the quantum level.

B. Beyond the Semi-Classical Level: Gaussian Approximation

Going beyond the semi-classical or 1st order moment approximation, we need to quantize the theory at least at the next leading order. If we consider the 2nd order central moment quantum fluctuation are the most important next leading order contribution (compared with higher order moment), we call it the Gaussian approximation or 2nd order central moment approximation, while the higher order moment are all called non-Gaussian fluctuations which might be important near local singularities of the spacetime when local phase transition happens, although the intrinsic global singularity can be avoided by guaranteeing the global homotopy group trivial.

At the Gaussian approximation, $\delta g^{(2)}_{\mu\nu}$ can be given by a perturbative one-loop calculation [37, 38] of the NLSM when it is relatively small compared with $g^{(1)}_{\mu\nu}$

$$\delta g^{(2)}_{\mu\nu}(X) = \frac{R^{(1)}_{\mu\nu}(X)}{32\pi^2\lambda}\delta k^2, \tag{9}$$

where $R^{(1)}_{\mu\nu}$ is the Ricci curvature given by 1st order metric $g^{(1)}_{\mu\nu}$, k^2 is the cutoff energy scale of the Fourier component of the frame fields. The validity of the perturbation calculation $R^{(1)}\delta k^2 \ll \lambda$ is the validity of the Gaussian approximation, which can be seen as follows. It will be shown in later section that λ is nothing but the critical density ρ_c of the universe, $\lambda \sim O(H_0^2/G)$, H_0 the Hubble's constant, G the Newton's constant. Thus for our concern of pure gravity in which matter is ignored, the condition $R^{(1)}\delta k^2 \ll \lambda$ is equivalent to $\delta k^2 \ll 1/G$ which is reliable except for some local singularities are developed when the Gaussian approximation is failed.

The equation (9) is nothing but a RG equation or known as the Ricci flow equation (some reviews see e.g. [41–43])

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2R_{\mu\nu}, \tag{10}$$

with flow parameter $\delta t = -\frac{1}{64\pi^2\lambda}\delta k^2$ having dimension of length squared $[L^2]$, which continuously deform the spacetime metric driven by its Ricci curvature.

For the Ricci curvature is non-linear for the metric, the Ricci flow equation is a non-linear version of a heat equation for the metric, and flow along t introduces an averaging or coarse-graining process to the intrinsic non-linear gravitational system which is highly non-trivial [44–48]. In general, if the flow is free from local singularities there exists long flow-time solution in $t \in (-\infty, 0)$, which is often called ancient solution in mathematical literature. This range of the t-parameter corresponds to $k \in (0, \infty)$, that is from $t = -\infty$, i.e. the short distance (high energy) UV scale $k = \infty$ forwardly to $t = 0$ i.e. the long distance (low energy) IR scale $k = 0$. The metric at certain scale t is given by being averaged out the shorter distance details which produces an effective correction to the metric at that scale. So along t, the manifolds loss its information in shorter distance, thus the flow is irreversible, i.e. generally having no backwards solution, which is the underlying reason for the non-unitary and existence of entropy of a spacetime.

As it is shown in (4), (5), the 2nd order moment fluctuation modifies the local (quadratic) distance of the spacetime, so the flow is non-isometry. This is an important feature worth stressing, which is the underlying reason for the anomaly. The non-isometry is not important for its topology, so along t, the flow preserves the topology of the spacetime but its local metric, shape and size (volume) changes. There also exists a very special solution of the Ricci flow called Ricci Soliton, which only changes the local volume while keeps its local shape. The Ricci Soliton, and its generalized version, the Gradient Ricci Soliton, as the flow limits, are the generalization of the notion of fixed point in the sense of RG flow. The Ricci Soliton is an important notion for understanding the gravity at cosmic scale and studying the the thermodynamics of the Ricci flow at equilibrium.

The Ricci flow was initially introduced in 1980s by Friedan [34, 35] in $d = 2 + \epsilon$ NLSM and independently by Hamilton in mathematics [49, 50]. The main motivation of introducing it from the mathematical point of view is to classify manifolds, a specific goals is to proof the Poincare conjecture. Hamilton used it as a useful tool to gradually deform a manifolds into a more and more "simple and good" manifolds whose topology can be readily recognized for some simple cases. A general realization of the program is achieved by Perelman at around 2003 [3, 51, 52], who introduced several monotonic functionals to successfully deal with the local singularities which might be developed in more general cases. The Ricci flow approach is not only powerful to the compact geometry (as Hamilton's and Perelman's seminal works had shown) but also to the non-compact [53–55] and the Lorentzian geometry [15, 56–62].

C. The Wavefunction and Density Matrix at the Gaussian Approximation

So far we have not explicitly defined the quantum state of the reference frame $|X\rangle$ in (1). In fact, the previous (2nd order) results e.g. (5), (9) and hence the Ricci flow (10) can also equivalently be given by the expectation value $\langle O\rangle = \langle X|O|X\rangle$ via explicitly writing down the wavefunction $\Psi(X)$ of the frame fields at the Gaussian approximation. Note that at the semi-classical level, the frame fields X is a delta-distribution and peaks at its mean value, and further more, the action of the NLSM seems like a collection of harmonic oscillators, thus at the Gaussian approximation level, finite Gaussian width/2nd moment fluctuation of X must be introduced. When one performs a canonical quantization to the NLSM at the Gaussian approximation level, the fundamental solution of the wave function(al) (as a functional of the frame fields X^μ) of NLSM takes the Gaussian form, i.e. a coherent state

$$\Psi[X^\mu(x)] = \frac{1}{\sqrt{\lambda}(2\pi)^{D/4}} \frac{|\det \sigma_{\mu\nu}|^{1/4}}{|\det g_{\mu\nu}|^{1/4}} \exp\left[-\frac{1}{4}\left|X^\mu(x)\sigma_{\mu\nu}(x)X^\nu(x)\right|\right], \tag{11}$$

where the covariant matrix $\sigma_{\mu\nu}(x)$, playing the role of the Gaussian width, is the inverse of the 2nd order central moment fluctuations of the frame fields at point x

$$\sigma_{\mu\nu}(x) = \frac{1}{\sigma^{\mu\nu}(x)} = \frac{1}{\langle\delta X^\mu(x)\delta X^\nu(x)\rangle}, \tag{12}$$

which is also given by perturbative one-loop calculation up to a diffeomorphism of X. The absolute symbol of $|X^\mu\sigma_{\mu\nu}X^\nu|$ in the exponential is used to guarantee the quadratic form and hence the determinant of $\sigma_{\mu\nu}$ induced from the Gaussian integral over X positive even in the Lorentzian signature.

We can also define a dimensionless density matrix corresponding to the fundamental solution of the wavefunction

$$u[X^\mu(x)] = \Psi^*(X)\Psi(X) = \frac{1}{\lambda(2\pi)^{D/2}} \frac{\sqrt{|\det \sigma_{\mu\nu}|}}{\sqrt{|\det g_{\mu\nu}|}} \exp\left[-\frac{1}{2}\left|X^\mu(x)\sigma_{\mu\nu}X^\nu(x)\right|\right], \tag{13}$$

and $\frac{1}{\lambda(2\pi)^{D/2}}\frac{\sqrt{|\det\sigma_{\mu\nu}|}}{\sqrt{|\det g_{\mu\nu}|}}$ is a normalization parameter, so that

$$\lambda\int d^DX\Psi^*(X)\Psi(X)=\lambda\int d^DXu(X)=1, \tag{14}$$

in which we often attribute the flow of the volume form d^DX_t to the flow of the metric g_t, for the volume element $d^DX_t\equiv dV_t(X^\mu)\equiv\sqrt{|g_t|}dX^0dX^1dX^2dX^3$. Then the expectation values $\langle O\rangle$ can be understood as $\lambda\int d^DX_tuO$. As the quantum frame fields X are q-number in the theory, precisely speaking, the integral of them should be, in principle, a functional integral. Here the formal c-number integral of them $\int d^DX_t$... is for the conventional in the Ricci flow literature, in which X is a coarse-grained c-number coordinates of manifolds at scale t. The exact functional integral of X is considered in calculating the partition function and related anomaly of the theory in section-III.

Under a diffeomorphism of the metric, the transformation of $u(X)$ is given by a diffeomorphism of the covariant matrix (h is certain function)

$$\sigma_{\mu\nu}\to\hat{\sigma}_{\mu\nu}=\sigma_{\mu\nu}+\nabla_\mu\nabla_\nu h. \tag{15}$$

So there exists an arbitrariness in the density $u(X)$ for different choices of a diffeomorphism/gauge.

According to the statistical interpretation of wavefunction with the normalization condition (14), $u(X^0,X^1,X^2,X^3)$ describes the probability density that finding these frame particles in the volume $dV_t(X^\mu)$. As the spacetime X flows along t, the volume ΔV_t, in which density is averaged, also flows, so the density at the corresponding scale is coarse-grained. If we consider the volume of the lab, i.e the base space, is rigid and fixed by $\lambda\int d^4x=1$, by noting (14), we have

$$u[X^\mu(x),t]=\frac{d^4x}{d^DX_t}=\lim_{\Delta V_t\to 0}\frac{1}{\Delta V_t}\int_{\Delta V_t}1\cdot d^4x. \tag{16}$$

We can see that the density $u(X,t)$ can be interpreted as a coarse-grained density in the volume element $\Delta V_t\to 0$ w.r.t. a fine-grained unit density in the lab volume element d^4x at UV $t\to-\infty$.

In this sense, the coarse-grained density $u(X,t)$ is in analogy with the Boltzmann's distribution function, so it should satisfy an analogous irreversible Boltzmann's equation, and giving rise to an analogous Boltzmann's monotonic H-functional. In the following sections, we will deduce such equation and the functional of $u(X,t)$. The coarse-grained density $u(X,t)$ has profound physical and geometric meaning, it also plays a central role in analyzing the statistic physics of the frame fields and generalizes the manifolds to the density manifolds.

D. Ricci-DeTurck Flow

In previous subsection, from the viewpoint of frame fields particle, $u(X^\mu,t)$ has a coarse-grained particle density interpretation, the eq.(16) can also be interpreted as a manifolds density [63] from the geometric point of view. For instance, $u(X,t)$ associates a manifold density or density bundle to each point X of a manifolds, measures the fuzziness of the "point". It is worth stressing that the manifolds density $u(X,t)$ does not simply a conformal scaling of a metric by the factor, since if it is the case, the integral measure of $D=4$-volume or 3-volume in the expectation $\langle O\rangle=\lambda\int d^DXuO$ would scale by different powers. There are various useful generalizations of the Ricci curvature to the density manifolds, a widely accepted version is the Bakry-Emery generalization [64]

$$R_{\mu\nu}\to R_{\mu\nu}-\nabla_\mu\nabla_\nu\log u, \tag{17}$$

which is also used in Perelman's seminal paper. The density normalized Ricci curvature is bounded from below

$$R_{\mu\nu}-\nabla_\mu\nabla_\nu\log u\geq\sigma_{\mu\nu}, \tag{18}$$

if the density manifolds has finite volume.

As a consequence, replacing the Ricci curvature by the density normalized one, we get the Ricci-DeTurck flow [65]

$$\frac{\partial g_{\mu\nu}}{\partial t}=-2\left(R_{\mu\nu}-\nabla_\mu\nabla_\nu\log u\right), \tag{19}$$

which is equivalent to the standard Ricci flow equation (10) up to a diffeomorphism. Mathematically, the Ricci-DeTurck flow has the advantage that it turns out to be a gradient flow of some monotonic functionals introduced by Perelman, which have profound physical meanings shown later.

The eq.(14) and (16) also give a volume constraint to the fiducial spacetime (the lab), the coarse-grained density $u(X,t)$ cancels the flow of the volume element $\sqrt{|\det g_{\mu\nu}|}$, so

$$\frac{\partial}{\partial t}\left(u\sqrt{|\det g_{\mu\nu}|}\right)=0. \tag{20}$$

Together with the Ricci-DeTurck flow equation (19), we have the flow equation of the density

$$\frac{\partial u}{\partial t}=\left(R-\Delta_X\right)u, \tag{21}$$

which is in analogy to the irreversible Boltzmann's equation for his distribution function. Δ_X is the Laplacian operator in terms of the manifolds coordinates X. Note the minus sign in front of the Laplacian, it is a backwards heat-like equation. Naively speaking, the solution of the backwards heat flow will not exist. But we could also note that if one let the Ricci flow flows to certain IR scale t_*, and at t_* one might then choose an appropriate $u(t_*)=u_0$ arbitrarily (up to a diffeomorphism gauge) and flows it backwards in $\tau=t_*-t$ to obtain a solution $u(\tau)$ of the backwards equation. Now since the flow is consider free from global singularities for the trivialness of the homotopy group, we could simply choose $t_*=0$, so we defined

$$\tau=-t=\frac{1}{64\pi^2\lambda}k^2\in(0,\infty). \tag{22}$$

In this case, the density satisfies the heat-like equation

$$\frac{\partial u}{\partial \tau}=\left(\Delta_X-R\right)u, \tag{23}$$

which does admit a solution along τ, often called the conjugate heat equation in mathematical literature.

So far (23) together with (19) the mathematical problem of the Ricci flow of a Riemannian/Lorentzian manifolds is transformed to a coupled equations

$$\begin{cases}\frac{\partial g_{\mu\nu}}{\partial t}=-2\left(R_{\mu\nu}-\nabla_\mu\nabla_\nu\log u\right)\\ \frac{\partial u}{\partial \tau}=\left(\Delta_X-R\right)u\\ \frac{d\tau}{dt}=-1\end{cases} \tag{24}$$

and the manifolds (M^D,g) is generalized to a density manifolds (M^D,g,u) [63, 66, 67] with the constraint (14).

III. THE ANOMALY AND ITS IMPLICATIONS

At the semi-classical approximation, see in eq.(8), when the quantum fluctuations of the frame fields or spacetime coordinates are ignored, the general coordinates transformation is just a classical coordinates transformation. We will show that when the quantum fluctuations are taken into account in the general coordinates transformation beyond the semi-classical approximation, quantum anomaly emerges. As is seen in the previous section, the quantum fluctuation and hence the coarse-graining process of the Ricci flow does not preserve the quadratic distance of a geometry, see (4) and (5). The non-isometry of the quantum fluctuation induces a breakdown of diffeomorphism or general coordinate transformation at the quantum level, namely the diffeomorphism anomaly. In this section, we derive the diffeomorphism anomaly of the theory, show its relation to the Shannon entropy whose monotonicity gives an analogous H-theorem of the frame fields system and the Ricci flow. Further more, as the quantum frame fields theory describes a quantum spacetime, together with the generalized quantum Equivalence Principle, the anomaly induced effective action in terms of the Shannon entropy can also be interpreted as a gravity theory, which at low energy expansion is a scale dependent Einstein-Hilbert action plus a cosmological constant. This part has certain overlap with the previous work [2], for the self-containedness of the paper, we hope this section provide a general background and lay the foundation for the subsequent thermodynamic and statistic interpretation of the theory.

A. Diffeomorphism at the Quantum Level

First we consider the functional quantization of the pure frame fields without explicitly incorporating the matter source. The partition function is

$$Z(M^D)=\int[\mathcal{D}X]\exp\left(-S[X]\right)=\int[\mathcal{D}X]\exp\left(-\frac{1}{2}\lambda\int d^4xg^{\mu\nu}\partial_aX_\mu\partial_aX_\nu\right), \tag{25}$$

where M^D is the target spacetime, and the base space can be either Euclidean and Minkowskian. Since considering the action or the volume element $d^4x \equiv d^4x \det e$ ($\det e$ is a Jacobian) does not pick any imaginary i factor no matter the base space is in Minkowskian or Euclidean one, if one takes $dx_0^{(E)} \to idx_0^{(M)}$ then $\det e^{(E)} \to -i \det e^{(M)}$, so without loss of generality we use the Euclidean base spacetime in the following discussions, and remind that the result is the same for Minkowskian.

Note that a general coordinate transformation

$$X_\mu \to \hat{X}_\mu = \frac{\partial \hat{X}_\mu}{\partial X_\nu} X_\nu = e_\mu^\nu X_\nu \tag{26}$$

does not change the action $S[X] = S[\hat{X}]$, but the measure of the functional integral changes

$$\begin{aligned} \mathcal{D}\hat{X} &= \prod_x \prod_{\mu=0}^{D-1} d\hat{X}_\mu(x) = \prod_x \epsilon_{\mu\nu\rho\sigma} e_\mu^0 e_\nu^1 e_\rho^2 e_\sigma^3 dX_0(x) dX_1(x) dX_2(x) dX_3(x) \\ &= \prod_x |\det e(x)| \prod_x \prod_{a=0}^{D-1} dX_a(x) = \left(\prod_x |\det e(x)| \right) \mathcal{D}X, \end{aligned} \tag{27}$$

where

$$\epsilon_{\mu\nu\rho\sigma} e_\mu^0 e_\nu^1 e_\rho^2 e_\sigma^3 = \left| \det e_\mu^a \right| = \sqrt{|\det g_{\mu\nu}|} \tag{28}$$

is the Jacobian of the diffeomorphism. The Jacobian is nothing but a local relative (covariant basis) volume element $dV(\hat{X}_\mu)$ w.r.t. the fiducial volume $dV(X_a)$. Note that the normalization condition (14) also defines a fiducial volume element $ud^4X \equiv udV(\hat{X}_\mu)$, so the Jacobian is nothing but related to the frame fields density matrix

$$u(\hat{X}_\mu) = \frac{dV(X_a)}{dV(\hat{X}_\mu)} = |\det e_a^\mu| = \frac{1}{\left| \det e_\mu^a \right|}. \tag{29}$$

Here the absolute symbol of the determinant is because the density u and the volume element are kept positive defined even in the Lorentz signature. Otherwise, for the Lorentz signature, it should introduce some extra imaginary factor i into (30) to keep the condition (14). The density so defined followed by (14) is an explicit generalization from the standard 3-space density to a 4-spacetime version. It is the definition of the volume form and the manifolds density ensure the formalism of the framework formally the same with the Perelman's standard form even in the Lorentzian signature. The manifolds density encodes the most important information of a Riemannian or Lorentzian geometry, i.e. the local volume comparison.

In this case, if we parameterize a dimensionless solution u of the conjugate heat equation as

$$u(\hat{X}) = \frac{1}{\lambda(4\pi\tau)^{D/2}} e^{-f(\hat{X})}, \tag{30}$$

then the partition function $Z(M^D)$ is transformed to

$$\begin{aligned} Z(\hat{M}^D) &= \int [\mathcal{D}\hat{X}] \exp\left(-S[\hat{X}]\right) = \int \left(\prod_x |\det e| \right) [\mathcal{D}X] \exp\left(-S[X]\right) \\ &= \int \left(\prod_x e^{f + \frac{D}{2}\log(4\pi\tau)} \right) [\mathcal{D}X] \exp\left(-S[X]\right) \\ &= \exp\left(\lambda \int d^4x \left[f + \frac{D}{2} \log(4\pi\tau) \right] \right) \int [\mathcal{D}X] \exp\left(-S[X]\right) \\ &= \exp\left(\lambda \int_{\hat{M}^D} d^D X u \left[f + \frac{D}{2} \log(4\pi\tau) \right] \right) \int [\mathcal{D}X] \exp\left(-S[X]\right). \end{aligned} \tag{31}$$

Note that $N(\hat{M}^D)$ in the exponential of the change of the partition function

$$Z(\hat{M}^D) = e^{\lambda N(\hat{M}^D)} Z(M^D) \tag{32}$$

is nothing but a pure real Shannon entropy in terms of the density matrix u

$$N(\hat{M}^D)=\int_{\hat{M}^D}d^DXu\left[f+\frac{D}{2}\log(4\pi\tau)\right]=-\int_{\hat{M}^D}d^DXu\log u. \tag{33}$$

The classical action $S[X]$ is invariant under the general coordinates transformation or diffeomorphism, but the quantum partition function is no longer invariant under the general coordinates transformation or diffeomorphism, which is called diffeomorphism anomaly, meaning a breaking down of the diffeomorphism at the quantum level. The diffeomorphism anomaly is purely due to the quantum fluctuation and Ricci flow of the frame fields which do not preserve the functional integral measure and change the spacetime volume at the quantum level. The diffeomorphism anomaly has many profound consequences to the theory of quantum reference frame, e.g. non-unitarity, the trace anomaly, the notion of entropy, reversibility, and the cosmological constant.

The non-unitarity is indicated by the pure real anomaly term, which is also induced by the non-isometry or volume change, and consequently the non-invariance of the measure of the functional integral during the Ricci flow. Because of the real-defined volume form (29) for both Euclidean and Lorentzian signature, the pure real contribution of the anomaly and hence the non-unitarity are valid not only for spacetime with Euclidean but also for the Lorentzian signature, it is a rather general consequence of the Ricci flow of spacetime. Essentially speaking, the reason of the non-unitarity is because we have enlarged the Hilbert space of the reference frame, from a rigid classical frame to a fluctuating quantum frame. The non-unitarity implies the breakdown of the fundamental Schrodinger equation which is only valid on a classical time of inertial frame, the solution of which is in $\mathcal{H}_\psi$. A fundamental equation playing the role of the Schrodinger equation, which can arbitrarily choose any (quantum) physical system as time or reference frame, must be replaced by a Wheeler-DeWitt-like equation in certain sense [11], the solution of which is instead in $\mathcal{H}_\psi\otimes\mathcal{H}_X$. In the fundamental equation, the quantum fluctuation of physical time and frame, more generally, a general physical coordinates system must break the unitarity. We know that in quantum fields theory on curved spacetime or accelerating frame, the vacuum states of the quantum fields in difference diffeomorphism equivalent coordinate systems are unitarily inequivalent. The Unruh effect is a well known example: accelerating observers in the vacuum will measure a thermal bath of particles. The Unruh effect shows us how a general coordinates transformation (e.g. from an inertial to an accelerating frame) leads to the non-unitary anomaly (particle creation and hence particle number non-conservation), and how the anomaly will relate to a thermodynamics system (thermal bath). In fact, like the Unruh effect, the Hawking effect [68] and all non-unitary particle creation effects in a curved spacetime or accelerating frame are related to the anomaly in a general covariant or gravitational system. All these imply that the diffeomorphism anomaly will have deep thermodynamic interpretation which is the central issue of the paper.

Without loss of generality, if we simply consider the under-transformed coordinates X_μ identifying with the coordinates of the fiducial lab x_a which can be treated as a classical parameter coordinates, in this situation the classical action of NLSM is just a topological invariant, i.e. half the dimension of the target spacetime

$$\exp\left(-S_{cl}\right)=\exp\left(-\frac{1}{2}\lambda\int d^4xg^{\mu\nu}\partial_a x_\mu\partial_a x_\nu\right)=\exp\left(-\frac{1}{2}\lambda\int d^4xg^{\mu\nu}g_{\mu\nu}\right)=e^{-\frac{D}{2}}. \tag{34}$$

Thus the total partition function of the frame fields takes a simple form

$$Z(\hat{M}^D)=e^{\lambda N(\hat{M}^D)-\frac{D}{2}}. \tag{35}$$

B. The Trace Anomaly

The partition function now is non-invariance (32) under diffeomorphism at the quantum level, so if one deduces the stress tensor by $\langle\mathcal{T}_{\mu\nu}\rangle=-\frac{2}{\sqrt{|g|}}\frac{\delta\log Z}{\delta g^{\mu\nu}}$, its trace $\langle g^{\mu\nu}\rangle\langle\mathcal{T}_{\mu\nu}\rangle=0$ is difference from $\langle\mathcal{T}^\mu_\mu\rangle=\langle g^{\mu\nu}\mathcal{T}_{\mu\nu}\rangle$

$$\langle\Delta\mathcal{T}^\mu_\mu\rangle=\langle g^{\mu\nu}\rangle\langle\mathcal{T}_{\mu\nu}\rangle-\langle g^{\mu\nu}\mathcal{T}_{\mu\nu}\rangle=\lambda N(M^D) \tag{36}$$

known as the trace anomaly. Cardy conjectured [69] that in a $d=4$ theory, quantities like $\langle\mathcal{T}^\mu_\mu\rangle$ could be a higher dimensional generalization of the monotonic Zamolodchikov's c-function in $d=2$ conformal theories, leading to a suggestion of the a-theorem [70] in $d=4$ and other suggestions (e.g. [71, 72]). In the following subsections, we will show that the Shannon entropy N and generalized $\tilde{N}$ are indeed monotonic, which might have more advantages, e.g. suitable for a Lorentzian target spacetime and for general D.

Note that the Shannon entropy $N(M^D)$ can be expanded at small τ

$$\lambda N(\hat{M}^D)=\lambda\sum_{n=0}^{\infty}B_n\tau^n=\lambda\left(B_0+B_1\tau+B_2\tau^2+...\right)\quad(\tau\to 0). \tag{37}$$

For $D=4$ the first few coefficients are

$$B_0=\lim_{\tau\to 0}N=\frac{D}{2\lambda}\left[1+\log\left(\sqrt{\lambda}4\pi\tau\right)\right], \tag{38}$$

$$B_1=\lim_{\tau\to 0}\frac{dN}{d\tau}=\int_{\hat{M}^4}d^4X\sqrt{|g|}\left(R+\frac{D}{2\tau}\right), \tag{39}$$

$$B_2=\lim_{\tau\to 0}\frac{1}{2}\frac{d^2N}{d^2\tau}=-\int_{\hat{M}^4}d^4X\sqrt{|g|}\left|R_{\mu\nu}+\frac{1}{2\tau}g_{\mu\nu}\right|^2, \tag{40}$$

in which B_0 can be renormalized out, and a renormalized B_1 will contribute to the effective Einstein-Hilbert action of gravity, see following subsection D. And B_2, as a portion of the full anomaly, plays the role of the conformal/Weyl anomaly up to some total divergence terms, for instance, $\varDelta R$ terms and the Gauss-Bonnet invariant. That is, a non-vanishing B_2 term measures the broken down of the conformal invariance of $M^{D=4}$, otherwise, a vanishing B_2 means that the manifold is a gradient steady Ricci soliton as the fixed point of the Ricci-DeTurck flow, which preserves it shape (conformal invariant) during the flows.

We note that B_2 as the only dimensionless coefficient measures the anomalous conformal modes, in this sense, $N(M^D)$ indeed relates to certain entropy. However, since the conformal transformation is just a special coordinates transformation, thus it is clearly that the single B_2 coefficient does not measure the total (general coordinates transformation) anomalous modes. Obviously this theory at $2<d=4-\epsilon$ is not conformal invariant, thus as the theory flows along t, the degrees of freedom are gradually coarse-grained and hence the modes-counting should also change with the flow and the scale, as a consequence all coefficients B_n in the series and hence the total partition function $e^{\lambda N(M^D)}$ should measure the total anomalous modes at certain scale τ, leading to the full entropy and anomaly.

Different from some classically conformal invariant theories, e.g. the string theory, in which we only need to cancel a single scale-independent B_k coefficient in order to avoid conformal anomaly. As the theory at higher than 2-dimension is not conformal invariant, the full scale-dependent anomaly $N(M^D)$ is required to be canceled at certain scale. Fortunately, it will show in later subsection that a normalized full anomaly $\lambda\tilde{N}(M^D)$ can converge at UV for its monotonicity, thus giving rise to a finite counter term of order $O(\lambda)$ playing the role of a correct cosmological constant. The idea that the trace anomaly might have a relation to the cosmological constant is a recurring subject in literature [73–77], in the framework, the cosmological constant is naturally emerged in this way as the counter term of the trace anomaly (see subsection-D or [2]).

C. Relative Shannon Entropy and a H-Theorem for Non-Equilibrium Frame Fields

In the Ricci flow limit, i.e. the Gradient Shrinking Ricci Soliton (GSRS) configuration, the Shannon entropy N taking its maximum value N_*, it is similar with the thermodynamics system being in a thermal equilibrium state where its entropy is also maximal. In mathematical literature of Ricci flow, it is often defined a series of relative formulae w.r.t. the extreme values taking by the flow limit GSRS or analogous thermal equilibrium state denoted by a subscript *.

In GSRS, the covariance matrix $\sigma^{\mu\nu}$ as 2nd central moment of the frame fields with a IR cutoff k is simply proportional to the metric

$$\frac{1}{2}\sigma_*^{\mu\nu}=\frac{1}{2}\langle\delta X^\mu\delta X^\nu\rangle=\frac{1}{2\lambda}g^{\mu\nu}\int_0^{|p|=k}\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}=\frac{k^2}{64\pi^2\lambda}g^{\mu\nu}=\tau g^{\mu\nu}, \tag{41}$$

and then

$$\sigma_{*\mu\nu}=(\sigma_*^{\mu\nu})^{-1}=\frac{1}{2\tau}g_{\mu\nu}, \tag{42}$$

which means a uniform Gaussian broadening is achieved. And in this gauge, only longitudinal part of fluctuation exists.

When the density normalized Ricci curvature is completely given by the longitudinal fluctuation $\sigma_{\mu\nu}$, i.e. the inequality (18) saturates, giving a Gradient Shrinking Ricci Soliton (GSRS) equation

$$R_{\mu\nu}+\nabla_\mu\nabla_\nu f=\frac{1}{2\tau}g_{\mu\nu}. \tag{43}$$

It means, on the one hand, for a general $f(X)=\frac{1}{2}\left|\sigma_{\mu\nu}X^\mu X^\nu\right|$, so $R_{\mu\nu}$ seems vanish, so the standard Ricci flow equation (10) terminates; and on the other hand, the Ricci-DeTurck flow (19) only changes the longitudinal size or volume of the manifolds but its shape keep unchanged, thus the GSRS can also be seen stop changing, up to a size or volume rescaling. Thus the GSRS is a flow limit and can be viewed as a generalized RG fixed point.

In the following, we consider relative quantities w.r.t. the GSRS configuration. Considering a general Gaussian density matrix

$$u(X)=\frac{1}{\lambda(2\pi)^{D/2}}\frac{\sqrt{|\det\sigma_{\mu\nu}|}}{\sqrt{|\det g_{\mu\nu}|}}\exp\left(-\frac{1}{2}\left|X^\mu\sigma_{\mu\nu}X^\nu\right|\right), \tag{44}$$

in GSRS limit it becomes

$$u_*(X)=\frac{1}{\lambda(4\pi\tau)^{D/2}}\exp\left(-\frac{1}{4\tau}|X|^2\right). \tag{45}$$

Therefore, in GSRS, a relative density can be defined by the general Gaussian density $u(X)$ relative to the density $u_*(X)$ in GSRS

$$\tilde{u}(X)=\frac{u}{u_*}. \tag{46}$$

By using the relative density, a relative Shannon entropy $\tilde{N}$ can be defined by

$$\tilde{N}(M^D)=-\int d^DX\tilde{u}\log\tilde{u}=-\int d^DXu\log u+\int d^DXu_*\log u_*=N-N_*=-\log Z_P\leq 0, \tag{47}$$

where Z_P is nothing but the Perelman's partition function

$$\log Z_P=\int_{M^D}d^DXu\left(\frac{D}{2}-f\right)\geq 0, \tag{48}$$

and N_* is the maximum Shannon entropy

$$N_*=-\int d^DXu_*\log u_*=\int d^DXu_*\frac{D}{2}\left[1+\log(\sqrt{\lambda}4\pi\tau)\right]=\frac{D}{2\lambda}\left[1+\log(\sqrt{\lambda}4\pi\tau)\right]. \tag{49}$$

Since the relative Shannon entropy and the anomaly term is pure real, so the change of the partition function under diffeomorphism is non-unitary. For the coarse-graining nature of the density u, it is proved that the relative Shannon entropy is monotonic non-decreasing along the Ricci flow (along t),

$$\frac{d\tilde{N}(\hat{M}^D)}{dt}=-\tilde{\mathcal{F}}\geq 0, \tag{50}$$

where $\tilde{\mathcal{F}}=\mathcal{F}-\mathcal{F}_*\leq 0$ is the GSRS-normalized F-functional of Perelman

$$\mathcal{F}=\frac{dN}{d\tau}=\int_{M^D}d^DXu\left(R+|\nabla f|^2\right) \tag{51}$$

with the maximum value (at GSRS limit)

$$\mathcal{F}_*\equiv\mathcal{F}(u_*)=\frac{dN_*}{d\tau}=\frac{D}{2\lambda\tau}. \tag{52}$$

The inequality (50) gives an analogous H-theorem to the non-equilibrium frame fields and the irreversible Ricci flow. The entropy is non-decreasing along the Ricci flow making the flow irreversible in many aspects similar with the processes of irreversible thermodynamics, meaning that as the observation scale of the spacetime flows from short to long distance scale, the process losses information and the Shannon entropy increases. The equal sign in (50) can be taken when the spacetime configuration has flowed to a limit known as a Gradient Shrinking Ricci Soliton (GSRS), when the Shannon entropy takes its maximum value. Similarly, at the flow limit the density matrix u_* eq.(45) takes the analogous standard Maxwell-Boltzmann distribution.

D. Effective Gravity at Cosmic Scale and the Cosmological Constant

In terms of the relative Shannon entropy, the total partition function (35) of the frame fields is normalized by the GSRS extreme value

$$Z(M^D) = \frac{e^{\lambda N - \frac{D}{2}}}{e^{\lambda N_*}} = e^{\lambda \tilde{N} - \frac{D}{2}} = Z_P^{-\lambda} e^{-\frac{D}{2}} = \exp\left[\lambda \int_{M^D} d^D X u \left(f - D\right)\right]. \tag{53}$$

The relative Shannon entropy $\tilde{N}$ as the anomaly vanishes at GSRS or IR scale, however, it is non-zero at ordinary lab scale up to UV where the fiducial volume of the lab is considered fixed $\lambda \int d^4x = 1$. The cancellation of the anomaly at the lab scale up to UV is physically required, which leads to the counter term $\nu(M^D_{\tau=\infty})$ or cosmological constant. The monotonicity of $\tilde{N}$ eq.(50) and the W-functional implies [3, 78]

$$\nu(M^D_{\tau=\infty}) = \lim_{\tau\to\infty} \lambda \tilde{N}(M^D, u, \tau) = \lim_{\tau\to\infty} \lambda \mathcal{W}(M^D, u, \tau) = \inf_{\tau} \lambda \mathcal{W}(M^D, u, \tau) < 0, \tag{54}$$

where $\mathcal{W}$, the Perelman's W-functional, is the Legendre transformation of $\tilde{N}$ w.r.t. τ^{-1},

$$\mathcal{W} \equiv \tau \frac{\partial \tilde{N}}{\partial \tau} + \tilde{N} = \tau \tilde{\mathcal{F}} + \tilde{N} = \frac{d}{d\tau}\left(\tau \tilde{N}\right). \tag{55}$$

In other words, the difference between the effective actions (relative Shannon entropies) at UV and IR is finite

$$\nu = \lambda(\tilde{N}_{UV} - \tilde{N}_{IR}) < 0. \tag{56}$$

Perelman used his analogies: the temperature $T \sim \tau$, the (relative) internal energy $U \sim -\tau^2 \tilde{\mathcal{F}}$, the thermodynamics entropy $S \sim -\mathcal{W}$, and the free energy $F \sim \tau \tilde{N}$, up to proportional balancing the dimensions on both sides of $\sim$, the equation (55) is in analogy to the thermodynamics equation $U - TS = F$. So in this sense the W-functional is also called the W-entropy. Whether the thermodynamic analogies are real and physical, or just pure coincidences, is an important issue discussed in the next sections.

In fact $e^\nu < 1$ (usually called the Gaussian density [79, 80]) is a relative volume or the reduced volume $\tilde{V}(M^D_{\tau=\infty})$ of the backwards limit manifolds introduced by Perelman, or the inverse of the initial condition of the manifolds density $u^{-1}_{\tau=0}$. A finite value of it makes an initial spacetime with unit volume from UV flow and converge to a finite $u_{\tau=0}$, and hence the manifolds finally converges to a finite relative volume/reduced volume instead of shrinking to a singular point at $\tau = 0$.

As an example, for a homogeneous and isotropic universe for which the sizes of space and time (with a "ball" radius a_τ) are on an equal footing, i.e. a late epoch FRW-like metric $ds^2 = a^2_\tau(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2)$, which is a Lorentzian shrinking soliton configuration. Note that the shrinking soliton equation $R_{\mu\nu} = \frac{1}{2\tau} g_{\mu\nu}$ it satisfies and its volume form (29) are independent to the signature, so it can be approximately given by a 4-ball value $\nu(B^4_\infty) \approx -0.8$ [1, 2].

So the partition function, which is anomaly canceled at UV and having a fixed-volume fiducial lab, is

$$Z(M^D) = e^{\lambda \tilde{N} - \frac{D}{2} - \nu}. \tag{57}$$

Since $\lim_{\tau\to 0} \tilde{N}(M^D) = 0$, so at small τ, $\tilde{N}(M^D)$ can be expanded by powers of τ

$$\begin{aligned}
\tilde{N}(M^D) &= \frac{\partial \tilde{N}}{\partial \tau}\tau + O(\tau^2) = \tau \tilde{\mathcal{F}} + O(\tau^2) \\
&= \int_{M^D} d^D X u_{\tau\to 0} \left[\left(R_{\tau\to 0} + |\nabla f_{\tau\to 0}|^2 - \frac{D}{2\tau}\right)\tau\right] + O(\tau^2) \\
&= \int_{M^D} d^D X u_0 R_0 \tau + O(\tau^2),
\end{aligned} \tag{58}$$

in which $\lambda \int d^D X u_{\tau\to 0} \tau \left|\nabla f_{\tau\to 0}\right|^2 = \frac{D}{2}$ (at GSRS) has been used.

For $D = 4$ and small τ, the effective action of $Z(M^4)$ can be given by

$$-\log Z(M^4) = S_{eff} \approx \int_{M^4} d^4 X u_0 \left(2\lambda - \lambda R_0 \tau + \lambda \nu\right) \quad (\text{small}\, \tau). \tag{59}$$

Considering $u_0 d^4X = \sqrt{|g_t|}dV = \sqrt{|g_t|}dX^0dX^1dX^2dX^3$ is the invariant volume element, and using (22) to replace t or τ by cutoff scale k, we have

$$S_{eff} = \int_{M^4} dV\sqrt{|g_k|}\left(2\lambda - \frac{R_0}{64\pi^2}k^2 + \lambda\nu\right) \quad (\mathrm{small\,k}). \tag{60}$$

The effective action can be interpreted as a low energy effective action of pure gravity. As the cutoff scale k ranges from the lab scale to the solar system scale ($k > 0$), the action must recover the well-tested Einstein-Hilbert (EH) action. But at the cosmic scale ($k \to 0$), we know that the EH action deviates from observations and the cosmological constant becomes important. In this picture, as $k \to 0$, the action leaving $2\lambda + \lambda\nu$ should play the role of the standard EH action with a limit constant background scalar curvature R_0 plus the cosmological constant, so

$$2\lambda + \lambda\nu = \frac{R_0 - 2\Lambda}{16\pi G}. \tag{61}$$

While at $k \to \infty$, $\lambda\tilde{N} \to \nu$, the action leaving only the fiducial Lagrangian $\frac{D}{2}\lambda = 2\lambda$ which should be interpreted as a constant EH action without the cosmological constant

$$2\lambda = \frac{R_0}{16\pi G}. \tag{62}$$

Thus we have the cosmological term

$$\lambda\nu = \frac{-2\Lambda}{16\pi G} = -\rho_\Lambda. \tag{63}$$

The action can be rewritten as an effective EH action plus a cosmological term

$$S_{eff} = \int_{M^4} dV\sqrt{|g_k|}\left(\frac{R_k}{16\pi G} + \lambda\nu\right) \quad (\mathrm{small\,k}), \tag{64}$$

where

$$\frac{R_k}{16\pi G} = 2\lambda - \frac{R_0}{64\pi^2}k^2, \tag{65}$$

which is nothing but the flow equation of the scalar curvature [43]

$$R_k = \frac{R_0}{1 + \frac{1}{4\pi}Gk^2}, \quad \text{or} \quad R_\tau = \frac{R_0}{1 + \frac{2}{D}R_0\tau}. \tag{66}$$

Since at the cosmic scale $k \to 0$, the effective scalar curvature is bounded by R_0 which can be measured by "Hubble's constant" H_0 at the cosmic scale,

$$R_0 = D(D-1)H_0^2 = 12H_0^2, \tag{67}$$

so λ is nothing but the critical density of the 4-spacetime Universe

$$\lambda = \frac{3H_0^2}{8\pi G} = \rho_c, \tag{68}$$

so the cosmological constant is always of order of the critical density with a "dark energy" fraction

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = -\nu \approx 0.8, \tag{69}$$

which is close to the observational value. The detail discussions about the cosmological constant problem and the observational effect in the cosmology, especially the modification of the Distance-Redshift relation leading to the acceleration parameter $q_0 \approx -0.68$ can be found in [1, 2, 12, 13].

If matters are incorporated into the gravity theory, consider the entangled system in $\mathcal{H}_\psi \otimes \mathcal{H}_X$ between the to-be-studied quantum system (matters) and the quantum reference frame fields system (gravity). 2λ term in eq.(8) is normalized by the Ricci flow, by using eq.(60) and eq.(65), a matter-coupled-gravity is emerged from the Ricci flow

$$\begin{aligned} S[\psi, X] &\overset{(2)}{\approx} \int dV\sqrt{|g_k|}\left[\frac{1}{2}g^{\mu\nu}\frac{\delta\psi}{\delta X^\mu}\frac{\delta\psi}{\delta X^\nu} - V(\psi) + 2\lambda - \frac{R_0}{64\pi^2}k^2 + \lambda\nu\right] \\ &= \int dV\sqrt{|g_k|}\left[\frac{1}{2}g^{\mu\nu}\frac{\delta\psi}{\delta X^\mu}\frac{\delta\psi}{\delta X^\nu} - V(\psi) + \frac{R_k}{16\pi G} + \lambda\nu\right] \end{aligned} \tag{70}$$

IV. THERMAL EQUILIBRIUM STATE

A Gradient Shrinking Ricci Soliton (GSRS) configuration as a Ricci flow limit extremizes the Shannon entropy N and the W-functional. Similarly, a thermal equilibrium state also extremizes the H-functional of Boltzmann and the thermodynamic entropy. Thus the process of a generic Ricci flow flows into a GSRS limit is in analogy with the non-equilibrium state evolves into a thermal equilibrium state, they are not merely similar but even equivalent, when the thermal system is nothing but the frame fields system. In this section, following the previous discussions on the non-equilibrium state of the frame fields in 4-dimension, in a proper choice of time, we will discuss the thermal equilibrium state of the frame particle system as a GSRS configuration in lower 3-dimension, in which the temperature and several thermodynamic functions of the system can be explicitly calculated and the manifolds density can be interpreted as the thermal ensemble density of the frame fields particles, giving a statistical interpretation to Perelman's thermodynamic analogies of the Ricci flow.

A. A Temporal Static Shrinking Ricci Soliton as a Thermal Equilibrium State

When the shrinking Ricci soliton M^4 is static in the temporal direction, i.e. being a product manifolds $M^4 = M^3 \times \mathbb{R}$ and $\delta\mathbf{X}/\delta X_0 = 0$, where $X_0 \in \mathbb{R}$ is the physical time, $\mathbf{X} = (X_1, X_2, X_3) \in M^3$ is a 3-space gradient shrinking Ricci soliton of lower dimensions, we can prove here that the temporal static spatial part M^3 is in thermal equilibrium with the flow parameter τ proportional to its temperature, and the manifolds density u of M^3 can be interpreted as the thermal equilibrium ensemble density.

According to Masubara's formalism of thermal fields theory, the thermal equilibrium of the spatial frame fields can be defined by the periodicity $\mathbf{X}(\mathbf{x}, 0) = \mathbf{X}(\mathbf{x}, \beta)$ in their Euclidean time of the lab (remind that we start from the Euclidean base space for the frame fields theory), where $\beta = 1/T$ is the inverse of the temperature. Now the frame fields is a mapping $\mathbb{R}^3 \times S^1 \to M^3 \times \mathbb{R}$. Then in such configuration, the τ parameter of the 3-space shrinking soliton M^3 becomes

$$\tau = \frac{1}{2\lambda}\int \frac{d^3\mathbf{p}d\omega_n}{(2\pi)^4}\frac{1}{\mathbf{p}^2+\omega_n^2} = \frac{1}{2\lambda}T\sum_n\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\mathbf{p}^2+(2\pi nT)^2}, \tag{71}$$

where $\omega_n = 2\pi nT$, $\int \frac{d\omega_n}{2\pi} = T\sum_n$ have been used. The calculation is a periodic-Euclidean-time version of the general eq.(41). Since the density matrix eq.(45) of the frame fields X_μ is Gaussian or a coherent state, which the oscillators are almost condensed in the central peak, thus $\omega_0 = 0$ dominants the Masubara sum,

$$\tau = \frac{1}{2\lambda}T\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\mathbf{p}^2}. \tag{72}$$

Different from the naive notion of "temporal static" at the classical level, which means w.r.t. the physical clock X_0 of the quantum reference frame, i.e. $\langle\frac{\delta\mathbf{X}}{\delta X_0}\rangle = 0$. However, the notion "temporal static" is a little subtle at the quantum level. Because there is no "absolute static" at the quantum or microscopic level, since at such microscopic scale the modes are always in motion or vibrating w.r.t. the infinitely precise lab time x_0, i.e. $\frac{\partial\mathbf{X}(x)}{\partial x_0} \neq 0$. Actually $\partial\mathbf{X}/\partial x_0$ is in general non-zero even though its oscillation degrees of freedom are almost frozen (Masubara frequency ω_n is zero for the Gaussian wavefunction), while the center of the Gaussian wave pocket of $\mathbf{X}$ is in translational motion so $\mathbf{p} \neq 0$, so its expectation value is in general finite, for instance, $\langle\frac{\partial\mathbf{X}(x)}{\partial x_0}\rangle \sim \frac{3}{2}T < \infty$ claimed by the equipartition energy of the translational motion in 3-space. In general, whether or not the modes of the spatial frame fields is temporal static depends on the scale to evaluate the average of the physical clock $\langle X_0\rangle$. The notion of "thermal static" in the sense of statistical physics is approximate at a macroscopic scale rather than a microscopic scale, at which scale the molecules are always in motion (so does the physical clock X_0). The macroscopic scale of the thermal static system is at such a long physical time scale $\delta\langle X_0\rangle \gg \delta x_0$ that the averaged physical clock is almost frozen $\frac{\partial x_0}{\partial\langle X_0\rangle} \to 0$ w.r.t. the infinitely precise lab time x_0, so that the thermal static condition $\langle\frac{\delta\mathbf{X}}{\delta X_0}\rangle = \langle\frac{\partial\mathbf{X}}{\partial x_0}\rangle \cdot \frac{\partial x_0}{\partial\langle X_0\rangle} \to 0$ can be achieved.

More precisely, when we mention that the 3-space is macroscopic "temporal static", a IR cutoff, for example, H_0 as a macroscopic Hubble scale should be taken into account. The fluctuation modes on the 3-space outside the Hubble scale $0 < |\mathbf{p}| < H_0$ are frozen and temporal static, while those modes $|\mathbf{p}| > H_0$ inside the Hubble horizon are dynamic. So with this cutoff scale we have

$$\tau = \frac{1}{2\lambda}T\int_0^{|\mathbf{p}|=H_0}\frac{d^3\mathbf{p}}{(2\pi)}\frac{1}{\mathbf{p}^2} = \frac{C_3}{2\lambda}TH_0 = \frac{1}{\lambda_3}T = \frac{1}{\lambda_3\beta}, \tag{73}$$

where the 3-space energy density is $\lambda_3 = \frac{\lambda}{\frac{1}{2}C_3 H_0} = \frac{12\pi^2\lambda}{H_0}$. Note that if we consider the temporal integral is also cutoff at about a long physical time scale, e.g. the age of the universe $O(1/H_0)$, let the temporal direction is normalized as $\frac{1}{12\pi^2}\int_0^{12\pi^2/H_0} dx_0 H_0 = 1$, then the condition $\int d^4x\lambda \equiv 1$ gives its 3-space version

$$\int d^3x\lambda_3 = 1, \tag{74}$$

which is the definition of λ_3 on 3-space slice generalizing the critical density λ in a 4-spacetime covariant theory.

It is worth stressing that since the spatial slice depends on the definition of time, so the value of λ_3 is not universal (not necessarily equal to above $\frac{12\pi^2\lambda}{H_0}$ in other frame or cutoff, unlike the universal 4-spacetime critical density λ) but frame dependent. If a specific gauge of time or frame is chosen, λ_3 could be considered fix and be used as a proportional to correlate the τ parameter with the temperature of the temporal static frame fields configuration in such a specific gauge of time. The 3-space energy density λ_3 is very useful when we consider a temporal static GSRS spacetime or corresponding thermal equilibrium frame fields ensemble in later discussions.

In summary, an important observation is that when M^3 is a shrinking Ricci soliton in a temporal static product shrinking soliton $M^3 \times \mathbb{R}$, the global τ parameter of M^3 can be interpreted as a thermal equilibrium temperature defined by the Euclidean time periodic of the frame fields, up to a proportional being a 3-space energy density λ_3 (satisfying eq.(74)) balancing the dimensions between τ and T. Since temperature T is frame dependent, so is the proportional λ_3. The observation gives us a reason why in Perelman's paper τ could be analogous to the temperature T. The same results can also be obtained if one use the Lorentzian signature for the lab or base spacetime of the frame fields theory (2). In this case the thermal equilibrium of the spatial frame fields instead are subject to periodicity in the imaginary Minkowskian time $\mathbf{X}(\mathbf{x},0) = \mathbf{X}(\mathbf{x},i\beta)$, but even though the base spacetime is Wick rotated, the path integral does not pick any imaginary i factor in front of the action in (25) as the starting point, so the main results of the discussions retain independent to the signature of the base spacetime.

B. Thermodynamic Functions

For the thermodynamic interpretation of the quantum reference frame and gravity theory, in this subsection, we derive other thermodynamic functions of the system beside the temperature in the previous subsection, which are similar with the ideal gas. So the frame fields system in the Gaussian approximation can be seen as a system of frame fields gas, which manifests a underlying statistic picture of Perelman's thermodynamics analogies of his functionals. As convention, we all take the temperature $T = \lambda_3\tau$, eq.(73), $D = 3$ and λ replaced by λ_3, it is equivalent to choose a specific gauge of time for the thermal equilibrium frame fields configuration.

When the spatial shrinking soliton M^3 is in temporal static $dX_0 = 0$ and in thermal equilibrium, the partition function of the thermal ensemble of the frame fields $\mathbf{X}$ can be given by the trace/integration of the density matrix,

$$Z_*(\tau) = \lambda_3 \int d^3\mathbf{X} u(\mathbf{X}) = \lambda_3 \int d^3\mathbf{X} e^{-\frac{\mathbf{X}^2}{4\tau}} = \lambda_3(4\pi\tau)^{3/2}, \tag{75}$$

the normalized u density can be given by the 3-dimensional version of eq.(45)

$$u_*(\mathbf{X}) = \frac{1}{Z_*}u(\mathbf{X}) = \frac{1}{\lambda_3(4\pi\tau)^{3/2}} e^{-\frac{\mathbf{X}^2}{4\tau}}. \tag{76}$$

The partition function can also be consistently given by (35) with $D = 3$ in thermal equilibrium and hence the partition function of the frame fields in the shrinking soliton configuration

$$Z_*(\tau) = e^{\lambda_3 N_*(M^3) - \frac{3}{2}} = \exp\left[-\lambda_3 \int_{M^3} d^3X u_* \log u_* - \frac{3}{2}\right] = \lambda_3(4\pi\tau)^{3/2} = V_3\left(\frac{4\pi\lambda_3^{1/3}}{\beta}\right)^{3/2} = Z_*(\beta), \tag{77}$$

where $V_3 = \int d^3x$ is the 3-volume with the constraint $\lambda_3 V_3 = 1$. The partition function is identified with the partition function of the canonical ensemble of ideal gas (i.e. non-interacting frame fields gas in the lab) of temperature $1/\beta$ and gas particle mass $\lambda_3^{1/3}$. The interactions are effectively absorbed into the broadening of the density matrix and normalized mass of the frame fields gas particles.

The physical picture of frame fields gas in thermal equilibrium lays a statistical and physical foundation to Perelman's analogies between his functionals and thermodynamics equations as follows.

The internal energy of the frame fields gas can be given similar to the standard internal energy of ideal gas $\frac{3}{2}T$ given by the equipartition energy of translational motion in 3-space. Consider β as the Euclidean time of the flat lab, the internal energy seen from an observer in the lab is

$$E_* = -\frac{\partial \log Z_*}{\partial \beta} = \lambda_3^2\tau^2\frac{\partial N_*}{\partial \tau} = \lambda_3^2\tau^2\mathcal{F}_* = \frac{3}{2}\lambda_3\tau = \frac{3}{2}T, \tag{78}$$

in which (52) with $D=3$ and $\lambda\to\lambda_3$ have been used.

The fluctuation of the internal energy is given by

$$\langle E_*^2\rangle - \langle E_*\rangle^2 = \frac{\partial^2 \log Z_*}{\partial \beta^2} = \frac{3}{2}\lambda_3^2\tau^2 = \frac{3}{2}T^2. \tag{79}$$

The Fourier transformation of the density $u_*(\mathbf{X})$ is given by

$$u_*(\mathbf{K}) = \int d^3X u_*(\mathbf{X}) e^{-i\mathbf{K}\cdot\mathbf{X}} = e^{-\tau\mathbf{K}^2}, \tag{80}$$

since u satisfies the conjugate heat equation (23), so $\mathbf{K}^2$ is the eigenvalue of the Laplacian $-4\Delta_X + R$ of the 3-space, taking the value of the F-functional,

$$\mathbf{K}^2 = \lambda_3 \int d^3X \left(R|\Psi|^2 + 4|\nabla\Psi|^2\right) = \lambda_3\mathcal{F}, \tag{81}$$

so

$$u_*(\mathbf{K}^2) = e^{-\lambda_3\tau\mathcal{F}}. \tag{82}$$

For a state taking energy $\lambda_3^2\tau^2\mathcal{F} = E$, the probability density of the state can be rewritten as

$$u_*(E) = e^{-\frac{E}{\lambda_3\tau}} = e^{-\frac{E}{T}}, \tag{83}$$

which is the standard Boltzmann's probability distribution of the state. So we can see that the (Fourier transformed) manifolds density can be interpreted as the thermal equilibrium canonical ensemble density of the frame fields.

The free energy is given by

$$F_* = -\frac{1}{\beta}\log Z_* = -\lambda_3\tau\log Z_* = -\frac{3}{2}\lambda_3\tau\log(4\pi\tau), \tag{84}$$

similar with the standard free energy of ideal gas $-\frac{3}{2}T\log T$ up to a constant.

The minus H-functional of Boltzmann at an equilibrium limit and the thermal entropy of the frame fields gas can be given by the Shannon entropy

$$\lambda_3 N_* = S_* = -\lambda_3\int d^3X u_*\log u_* = \frac{3}{2}\left[1+\log(4\pi\tau)\right], \tag{85}$$

similar with the thermal entropy of fixed-volume ideal gas $\frac{3}{2}\log T + \frac{3}{2}$ up to a constant. The thermal entropy can also be consistently given by the standard formula

$$S_* = \log Z_* - \beta\frac{\partial \log Z_*}{\partial\beta} = \frac{3}{2}\left[1+\log(4\pi\tau)\right]. \tag{86}$$

which is in analogy with the fact that the W functional is the Legendre transformation of the relative Shannon entropy w.r.t. τ^{-1}. For this reason, the W functional is also an entropy function related to the (minus) thermodynamics entropy.

In summary, we have seen that, under general frame fields (coordinates) transformation the Shannon entropy anomaly N appearing in the partition function (32) (or relative Shannon entropy $\tilde{N}$ w.r.t. N_*) has profound thermodynamics interpretations. The Ricci flow of frame fields lead to non-equilibrium and equilibrium thermodynamics of the quantum spacetime, we summarize the comparisons between them in the Table I and II.

Frame fields at non-Ricci-flow-limit	Non-equilibrium thermodynamics
Relative Shannon entropy: $\tilde{N} = -\int d^3\mathbf{X}\tilde{u}(\mathbf{X},t)\log\tilde{u}(\mathbf{X},t)$	Boltzmanian H function: $H(t) = \int d^3\mathbf{v}\rho(\mathbf{v},t)\log\rho(\mathbf{v},t)$
Ricci flow parameter: t	Newtonian time: t
Monotonicity: $\frac{d\tilde{N}}{dt} = -\tilde{\mathcal{F}} \geq 0$	H theorem: $\frac{dH}{dt} \leq 0$
conjugate heat equation: $\frac{\partial u}{\partial t} = (-\Delta + R)\, u$	Boltzmann equation of ideal gas: $\frac{\partial \rho}{\partial t} = -\mathbf{v}\cdot\boldsymbol{\nabla}\rho$

Table I: Frame fields in general Ricci flow at non-flow-limit and the Non-equilibrium thermodynamics.

Frame fields at the Ricci flow limit (GSRS)	Equilibrium thermodynamics of ideal gas
partition function: $Z_*(\tau) = \lambda_3(4\pi\tau)^{3/2}$	partition function: $Z(T) = V_3(2\pi mT)^{3/2}$
GSRS flow parameter: $\lambda_3\tau$	temperature: $T = \beta^{-1}$
$\lambda_3^2\tau^2\mathcal{F}_* = \frac{3}{2}\lambda_3\tau$	internal energy: $E_* = -\frac{\partial \log Z}{\partial \beta} = \frac{3}{2}T$
manifold density: $u_*(\mathbf{K}) = e^{-\tau\mathbf{K}^2} = e^{-\lambda_3\tau\mathcal{F}}$	canonical ensemble density: $\rho = e^{-\frac{E}{T}}$
$-\lambda_3\tau\log Z_* = -\frac{3}{2}\lambda_3\tau\log(4\pi\tau)$	free energy: $F_* = -T\log Z(T) = -\frac{3}{2}T\log T$
Shannon entropy: $\lambda_3 N_* = \frac{3}{2}\left[1+\log(4\pi\tau)\right]$	thermodynamic entropy: $S_* = \frac{3}{2}\left(1+\log T\right)$
W functional: $\mathcal{W} = \tau\frac{d\tilde{N}}{d\tau} + \tilde{N}$	first law of thermodynamics: $E_* - TS_* = F_*$
Monotonicity: $\frac{d\tilde{N}}{dt} \geq 0$	second law of thermodynamics: $\delta S \geq 0$

Table II: Frame fields in Gradient Shrinking Ricci Soliton (GSRS) configuration and the equilibrium thermodynamics of ideal gas.

V. APPLICATION TO THE SCHWARZSCHILD BLACK HOLE

In this section, we try to apply the general statistic and thermodynamics interpretation of the quantum frame fields to a physical gravitational system, as one of the touchstone of quantum gravity, i.e. to understand the statistical origin of the thermodynamics of the Schwarzschild black hole.

A. The Temperature of a Schwarzschild Black Hole

The region in the vicinity of the origin of a Schwarzschild black hole is an example of classical static shrinking Ricci soliton. A rest observer distant from it sees an approximate metric $M^3\times\mathbb{R}$, where the region in the vicinity of the origin of the spatial part M^3 is a shrinking Ricci soliton. The reason is as follows, because the black hole satisfies the Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\mathcal{T}_{\mu\nu}, \tag{87}$$

where the stress tensor is a point distributed matter in rest with a mass m at the origin $x=0$ (seen from the distant rest observer)

$$\mathcal{T}_{00} = m\delta^{(3)}(\mathbf{x}), \quad \mathcal{T}_{ij} = 0 \quad (i,j=1,2,3), \tag{88}$$

where Latin index i,j is for spatial index in the following. So we have

$$R(\mathbf{x}) = -8\pi G\mathcal{T}^\mu_\mu = 8\pi Gm\delta^{(3)}(\mathbf{x}). \tag{89}$$

From the Einstein's equation we have the Ricci curvature of M^3 is proportional to the metric of M^3

$$R_{ij}(\mathbf{x}) = 8\pi G\mathcal{T}_{ij} + \frac{1}{2}g_{ij}R = \frac{1}{2}8\pi Gm\delta^{(3)}(\mathbf{x})g_{ij} \quad (i,j=1,2,3). \tag{90}$$

The equation is nothing but a normalized shrinking Ricci soliton equation (43) for M^3

$$R_{ij}(\mathbf{x}) = \frac{1}{2\tau}g_{ij}(\mathbf{x}) \quad (\mathbf{x}\approx 0) \tag{91}$$

with

$$\delta^{(3)}(\mathbf{x})\tau = \frac{1}{8\pi Gm}, \tag{92}$$

where $\delta^{(3)}(x)$ plays the role of the 3-space energy density λ_3 in the vicinity of the origin, satisfying $\int d^3x\delta^{(3)}(x) = 1$ as eq.(74), so by using the relation between τ and temperature T (73), we can directly read from the equation that a temperature seen by the lab's infinite distant rest observer is

$$T = \delta^{(3)}(\mathbf{x})\tau = \frac{1}{8\pi Gm}, \tag{93}$$

which is the standard Hawking's temperature of the Schwarzschild black hole seen by a distant rest observer.

Is the vacuum region outside the origin of the black hole also a shrinking Ricci soliton? One may naively think that the answer is no, since at the classical level, it seems $R_{ij} = 0$ (not a shrinking soliton eq.(91)), since outside the origin is just vacuum. But as is discussed in the next subsection, we argue that it is not true at the quantum level, if the vacuum and the vicinity region of the origin are in thermal equilibrium, they must be a shrinking Ricci solitons as a whole, i.e. $\langle R_{ij}\rangle = \frac{1}{2\tau}g_{ij} \neq 0$, eq.(101) in the "vacuum". The above result can be extended to the "vacuum" region outside the origin, the price to pay is that the "vacuum" is full of internal energy corresponding to the Hawking temperature. If the whole spacetime have not been in thermal equilibrium yet, the configuration has to irreversibly go on flowing to a common thermal equilibrium fixed point (a global shrinking Ricci soliton), leading to a global maximized entropy, as the H-theorem asserts.

B. The Energy of a Schwarzschild Black Hole

In classical general relativity, the mass m is often mentioned as the ADM energy of the black hole

$$m = \int d^3\mathbf{x}\mathcal{T}_{00} = \int d^3\mathbf{x} m\delta^{(3)}(\mathbf{x}), \tag{94}$$

seen by the distant rest observer (w.r.t. the lab time x_0). Here at the quantum level, the coordinates or frame fields and spacetime are quantum fluctuating, which gives rise to the internal energy related to the periodicity of the (Euclidean) lab time x_0 (i.e. $\beta = \frac{1}{T}$). So, mathematically speaking, the anomaly of the trace of the stress tensor will modify the total ADM mass at the quantum level, see (36). Since the anomaly of the action of the frame fields $\lambda_3 N_*$ representing the spacetime part is always real, the internal energy of the frame fields is given by the (78)

$$E_* = -\frac{\partial \log Z_*}{\partial\beta} = \frac{3}{2}T = \frac{3}{16\pi Gm}, \tag{95}$$

in which we have considered the 3-space volume V_3 outside the origin is in thermal equilibrium with the Hawking's temperature at the origin eq.(93), sharing the same equilibrium temperature T in the 3-volume V_3.

We can see that the internal energy E_* is an extra contribution to the total energy of the (black hole + "vacuum") system seen by the distant rest observer. Essentially this term can be seen as a quantum correction or a part of the trace anomaly contribution to the stress tensor, thus the total energy of the black hole including the classical ADM energy and the quantum fluctuating internal energy of the metric is

$$m_{BH} = \int d^3\mathbf{x}\,\langle\mathcal{T}_{00}\rangle = m + E_* = m + \frac{3}{16\pi G}\frac{1}{m}, \tag{96}$$

where the classical stress tensor $\mathcal{T}_{00}$ is formally replaced by its quantum expectation value

$$\langle\mathcal{T}_{00}\rangle = m\delta^{(3)}(\mathbf{x}) + \frac{3}{2}\frac{T}{V_3}. \tag{97}$$

A quantum Equivalence Principle should assert that the total energy rather than only the classical ADM mass contributes to the gravitation.

For a macroscopic classical black hole, $m \gg \sqrt{\frac{1}{G}}$, the first term ADM energy dominants the eq.(96),

$$m_{BH} \approx m. \tag{98}$$

The second internal energy term is gradually non-negligible for a microscopic quantum black hole. An important effect of the existence of the second term in (96) is, for a microscopic quantum black hole, it makes the total energy bound from below, the minimal energy is of order of the Planck mass

$$m_{BH} \geq \sqrt{\frac{3}{4\pi G}} \sim O(m_p), \tag{99}$$

which seems to prevent the black hole evaporating into nothing.

Further more, the internal energy $\frac{3}{2}T$ term contributing to the total energy m_{BH} and gravitation also demands that, not only the vicinity of the origin of the black hole is a shrinking soliton (as previous subsection claims), at the quantum level the whole 3-space is also the same shrinking soliton (i.e. satisfying eq.(91) with the identical τ globally and hence the same temperature T everywhere for the whole 3-space), just replacing the $\delta^{(3)}$-density in eq.(93) by the λ_3-density, which extends the $\delta^{(3)}$-density at the origin to the outside region (the "vacuum"), we have

$$T = \lambda_3 \tau = \frac{1}{8\pi G m}, \quad \text{with} \quad \int d^3\mathbf{x} \lambda_3 = \int d^3\mathbf{x} \frac{\langle \mathcal{T}_{00} \rangle}{m_{BH}} = 1 \tag{100}$$

for the whole thermal equilibrium 3-space, although at the classical level the vacuum $R_{ij}(x \neq 0) = 0$ is seem not a shrinking soliton outside the origin. The physical reason is transparent that the internal energy's contribution $\frac{3T}{2V_3}$ in $\langle \mathcal{T}_{00} \rangle$ also plays the role of an additional source of gravity outside the origin. For the whole 3-space with $\langle \mathcal{T}_{00} \rangle \neq 0$ and $\langle \mathcal{T}_{ij} \rangle = 0$, the Einstein's equation for the whole 3-space is nothing but the Shrinking Ricci Soliton equation (91):

$$\langle R_{ij} \rangle = \frac{1}{2} \langle R \rangle g_{ij} = \frac{1}{2} 8\pi G \langle \mathcal{T}_{00} \rangle g_{ij} \approx \frac{1}{2} 8\pi G m \frac{\langle \mathcal{T}_{00} \rangle}{m_{BH}} g_{ij} = \frac{1}{2T} \lambda_3 g_{ij} = \frac{1}{2\tau} g_{ij} \neq 0 \tag{101}$$

in which $\langle R \rangle = -8\pi G \langle \mathcal{T}_\mu^\mu \rangle = 8\pi G \langle \mathcal{T}_{00} \rangle \neq 0$ is used in the "vacuum" outside the origin. The equation is in fact the spatial components of the Gradient Shrinking Ricci Soliton equation (43) where $\langle R_{ij} \rangle = R_{ij} + \nabla_i \nabla_j f$, the Gaussian/thermal broadening of the density matrix u contributes to the classical curvature. The vicinity region of the origin plus the "vacuum" outside the origin of the black hole as a whole, is nothing but globally a shrinking Ricci soliton. The "vacuum" is not completely nothing at the quantum level but full of thermal particles $\langle \mathcal{T}_{00}(x \neq 0) \rangle \neq 0$. The Hawking temperature is essentially an Unruh effect, in certain sense, the Gradient Shrinking Ricci soliton equation, eq.(101), might play a more fundamental role than the Unruh's formula, which determines how local acceleration or gravitation gives rise to temperature.

The internal energy of the spacetime frame fields is an additional and necessary source of gravity, although macroscopically it is too small to contribute, at the quantum level its contribution is crucial for the 3-space in thermal equilibrium just right being a global shrinking Ricci soliton. The thermal internal energy coming from the quantum fluctuation of the 3-space gravitates normally as the quantum Equivalence Principle will assert. Otherwise, we have to face a paradox as follows. If we consider a frame x having $\mathcal{T}_{\mu\nu}(x) = 0$ everywhere, so according to the classical gravity $R_{\mu\nu}(x) = 0$ everywhere, if we transform it to anther accelerating frame x', one expects $\mathcal{T}_{\mu\nu}(x) \to \mathcal{T}'_{\mu\nu}(x') = 0$, and hence $R'_{\mu\nu}(x') = 0$ everywhere. However, according to the Equivalence Principle, in the accelerating frame x' one should feel equivalent gravity $R'_{\mu\nu}(x') \neq 0$. It is clearly something is missing, a new dimension of the Equivalence Principle must be considered. In order to solve the paradox and retain the Equivalence Principle, a quantum effect (actually the effect from the diffemorphism anomaly such as the trace anomaly or the Unruh effect) must be introduced so that the accelerating frame must be particles creating from the "vacuum" and be thermal, which plays the role of an equivalent gravitational source making $R'_{\mu\nu}(x') \neq 0$. The Hawking temperature in the internal energy term of eq.(96) is in essential the Unruh temperature playing such role. In this sense, the validity of the Equivalence Principle should be extended to the reference frame described by quantum state.

C. The Entropy of a Schwarzschild Black Hole

In the general framework, the entropy of the black hole comes from the uncertainty or quantum fluctuation moment of the frame fields given by the manifolds density u, more precisely, the thermalized black hole entropy is measured by the maximized Shannon entropy in terms of the probability distribution u of the frame fields in the background of the black hole. So in this subsection, we calculate the u density distributed around the Schwarzschild black hole and then evaluate the corresponding entropy as a measure of the black hole entropy. After a proper definition of a zero-point of the Shannon entropy, it gives a standard Bekenstein-Hawking entropy.

For an observer in the distant lab rest frame, the contributions to the temporal static u density around the black hole is two folds. Beside the thermal distribution u_* in the "vacuum" or bulk outside the black hole horizon, which

gives rise to the ideal gas entropy (85) as the background entropy, there is an additional $\tilde{u}$ density distribute mostly in a exterior thin shell near the horizon, and sparsely in the bulk outside the horizon, which we will focus on. The reason is as follows. Because $\tilde{u}$ density satisfies the conjugate heat equation (23) on the classical background of the black hole, since the classical scalar curvature $R = 0$ outside the horizon, and the temperature (equivalently the parameter τ and the mass) can be seen unchange for the thermalized black hole i.e. $\frac{\partial \tilde{u}}{\partial \tau} = 0$, thus the conjugate heat equation for $\tilde{u}$ is approximately given by the 4-Laplacian equation on the Schwarzschild black hole

$$\Delta_X \tilde{u}(X) = 0, \quad (|\mathbf{X}| \geq r_H). \tag{102}$$

Now the temporal static density $\tilde{u}(X)$ plays a similar role like a solution of the Klein-Gordon equation on the static background of the black hole. The approximation of the conjugate heat equation is equivalent to interpret the Klein-Gordon modes as a "first"-quantization probability density (not second-quantization fields). As is well-known, there are modes falling into the black hole horizon and hence disappearing from the outside observer's view. Just like the negative Klein-Gordon modes falling into the negative energy states below the groundstate. In a flat background, the amplitudes of the modes falling into and going out of the horizon are identical. So in the second-quantization, the negative mode falling into the horizon can be reinterpreted as a single anti-particle with positive energy modes going out of the horizon with the identical amplitude. However, in a curved background, for instance, the spacetime near the black hole horizon, the statement is no longer true. The two amplitudes differ from each other by a non-unitary equivalent factor. Thus the negative mode falling into the black hole horizon are no longer be reinterpreted as a single anti-particle mode going out, rather than multi-particles thermo-ensemble. At the situation, the density $\tilde{u}$ describes the ensemble density of modes going exterior the horizon $|\mathbf{X}| \geq r_H$ which can be seen by an outside observer.

By a routine calculation of the solution near the exterior black hole horizon resembling a Rindler metric as a starting point, we denote the solution $\tilde{u}_{\mathbf{k}}(\rho)$, in which $\mathbf{k}$ represents the Fourier component/momentum in the direction that are orthogonal to the direction of radius with $\rho = \log(r - r_H)$, r the radius, $r_H = 2Gm$ the radius of the horizon. The equation becomes

$$-\frac{\partial^2 \tilde{u}_{\mathbf{k}}}{\partial \rho^2} + \mathbf{k}^2 e^{2\rho} \tilde{u}_{\mathbf{k}} = \omega^2 \tilde{u}_{\mathbf{k}}, \tag{103}$$

where ω is the eigen-energy of the modes. By using a natural boundary condition that $\tilde{u}$ vanishes at infinity, we can see that each transverse Fourier mode $\tilde{u}_{\mathbf{k}}$ can be considered as a free 1+1 dimensional quantum field confined in a box, one wall of the box is at the reflecting boundary $\rho_0 = \log \epsilon_0$ where $\epsilon_0 \approx 0$, and the other wall of the box is provided by the potential

$$V(\rho) = \mathbf{k}^2 e^{2\rho}, \tag{104}$$

which becomes large $V(\rho) \gg 1$ at $\rho > -\log \mathbf{k}$. So we can approximate the potential by the second wall at $\rho_w = -\log \mathbf{k}$. So the length of the box is given by

$$\Delta \rho = \rho_w - \rho_0 = -\log(\epsilon_0 \mathbf{k}). \tag{105}$$

Thus the thickness of the horizon is about $\Delta r \sim e^{\Delta \rho} \sim \epsilon_0 \mathbf{k}$.

The density $\tilde{u}_{\mathbf{k}}(\rho)$ is located in the box $\rho \in (\rho_0, \rho_w)$. In other words, the solution of $\tilde{u}$ density is located mainly in a thin shell near the horizon $r \in (r_H, r_H + \epsilon_0 \mathbf{k})$. Furthermore, the modes $\mathbf{k}$ is assumed normal distributed (with a tiny width described by the parameter τ). In this picture, without solving the equation, we can approximately write down the natural solution as $\tilde{u}_{\mathbf{k}}(r) \overset{\tau \to 0}{\approx} \delta(|\mathbf{k}|)\delta(r - r_H)$, while for finite and small τ, we have a nearly Gaussian form

$$\tilde{u}_{\mathbf{k}}(r) \approx \delta(|\mathbf{k}|) \cdot \frac{1}{(4\pi\tau)^{1/2}} e^{-\frac{(r-r_H)^2}{4\tau}} \approx \frac{1}{(4\pi|\mathbf{k}|^2\tau)^{1/2}} e^{-\frac{(r-r_H)^2}{4\tau}}, \quad (r > r_H) \tag{106}$$

The exterior horizon solution can be considered as the standing wave solution as the superposition of the modes falling into and coming out of the black hole horizon. Then we have (up to a constant)

$$\log \tilde{u}_{\mathbf{k}}(r) \overset{r \sim r_H}{\approx} -\frac{1}{2} \log\left(|\mathbf{k}|^2 \tau\right). \tag{107}$$

A routine calculation of the relative Shannon entropy or W-functional gives the entropy of each k-mode in the limit in which the width τ is very small,

$$\begin{aligned}
\lambda_3 \tilde{N}(\tilde{u}_{\mathbf{k}}) &= -\lambda_3 \int d^3X \tilde{u}_{\mathbf{k}} \log \tilde{u}_{\mathbf{k}} \\
&= \delta(|\mathbf{k}|) \int_{r_H}^{\infty} 4\pi r^2 dr \frac{1}{(4\pi\tau)^{1/2}} e^{-\frac{(r-r_H)^2}{4\tau}} \frac{1}{2} \log\left(|\mathbf{k}|^2 \tau\right) \\
&\overset{\tau \simeq 0}{=} \delta(|\mathbf{k}|) \frac{1}{4} A \log\left(|\mathbf{k}|^2 \tau\right),
\end{aligned} \tag{108}$$

where $A = 4\pi r_H^2$ is the area of the horizon.

It is naturally to assume the momentum $\mathbf{k}$ in the horizon shell is homogeneous,

$$|\mathbf{k}| = |k_r| = |k_\perp|, \tag{109}$$

where k_r is the momentum in the radius direction and $k_\perp$ in the transverse directions on the horizon. When we integrate over all k-modes, we have the total relative Shannon entropy weakly depending to τ

$$\begin{aligned}
\lambda_3 \tilde{N}(\tilde{u}) &= \lambda_3 \int d^3\mathbf{k} \tilde{N}(\tilde{u}_{\mathbf{k}}) \\
&= \frac{1}{4} A \int \frac{d^2 k_\perp}{(2\pi)^2} \log\left(|k_\perp|^2 \tau\right) \int dk_r \delta(k_r) \\
&\approx \frac{1}{4} A \int_0^{1/\epsilon} \frac{2\pi k_\perp dk_\perp}{(2\pi)^2} \log\left(|k_\perp|^2 \tau\right) \\
&= \frac{1}{4} A \times \frac{1}{2\pi\tau} \left[-\frac{\tau}{2\epsilon^2}\left(1 - \log\frac{\tau}{\epsilon^2}\right)\right] \\
&\approx -\frac{A}{16\pi\epsilon^2},
\end{aligned} \tag{110}$$

in which the transverse momentum is effectively cut off at an inverse of a fundamental UV length scale ϵ^2.

The relative Shannon entropy gives an area law of the black hole entropy. To determine the UV length cutoff ϵ^2, we need to consider the scale at which the relative entropy is defined to be zero (not only the black hole is locally thermal equilibrium, but also the asymptotic background spacetime is globally thermal equilibrium), thus we need to consider the flow of the asymptotic background spacetime. A natural choice of a thermal equilibrium Ricci flow limit of the background spacetime (the black hole is embedded) is an asymptotic homogeneous and isotropic Hubble universe with scalar curvature $R_0 = D(D-1)H_0^2 = 12H_0^2$ at scale t_{UV} where we could consider and normalize the relative entropy to be zero (leaving only the background ideal gas entropy), since there is no information of the local shape distortions in such GSRS background because of the vanishing of its Weyl curvature, while the global curvature is non-zero which codes the information of its global volume shrinking. Under such definition, taking the normalized Shrinking Ricci soliton equation (43) and (22), we have

$$\tau_{UV} = -t_{UV} = \frac{D}{2R_0} = \frac{1}{64\pi^2\lambda} k_{UV}^2, \tag{111}$$

by using the critical density (68), which gives a natural cutoff corresponding to the scale t_{UV},

$$\epsilon^2 = k_{UV}^{-2} = \frac{1}{D\pi} G = \frac{1}{4\pi} G. \tag{112}$$

This is exactly the Planck scale, which is a natural cutoff scale induced from the Hubble scale H_0 and λ of the framework. However, it is worth stressing that the Planck scale is not the absolute fundamental scale of the theory, it only has meaning w.r.t. the asymptotic Hubble scale. The only fundamental scale of the theory is the critical density λ which is given by a combination of both the Planck scale and Hubble scale, but each individual Planck or Hubble scale does not have absolute meaning. The UV (Planck) cutoff scale could tend to infinity while the complementary (Hubble) scale correspondingly tends to zero (asymptotic flat background), keeping λ finite and fixed.

At this point, if we define a zero-relative-entropy for an asymptotic Hubble universe of scalar curvature R_0, then the black hole in this asymptotic background has a non-zero thermodynamic entropy

$$S = -\lambda_3 \tilde{N}(\tilde{u}) = \frac{A}{4G}, \tag{113}$$

up to the bulk background entropy $\lambda_3 N_* = S_* \ll S$, eq.(86). Combining the relative Shannon entropy $\tilde{N}$ and the bulk thermal background entropy N_*, and using the total partition function eq.(35), $Z(M^3) = e^{\lambda_3 N - \frac{3}{2}} = e^{\lambda_3(\tilde{N}+N_*) - \frac{3}{2}}$ we can also reproduce the total energy of the black hole in (96)

$$m_{BH} = -\frac{\partial \log Z}{\partial \beta} = m + \frac{3}{2} T, \tag{114}$$

in which eq.(47) and $A = 4\pi r_H^2 = 16\pi G^2 m^2 = \frac{\beta^2}{4\pi}$ have been used.

Different from the holographic idea that the information or entropy are coded in the (infinite thin and 2-dimensional) horizon or boundary of a gravitational system, in this framework where the coordinates of the spacetime geometry are smeared by quantum fluctuation, as a consequence that there is no mathematically precise notion of an infinite thin boundary in a "density manifolds" in general, it is just a semi-classical concept. Note that manifolds density u is mainly distributed at the horizon with a finite thickness (although very small), which contributes most of the anomaly and entropy to the black hole, so although the entropy (113) is proportional to the area, the geometric gravitational entropy given by the framework essentially comes from the 3-volume (note the 3d integral in eq.(108) and eq.(110)) but the 2-surface boundary. Or in other equivalent words, here the area of the horizon is fluctuating (due to its finite thickness) rather fixed, while the total energy and hence the temperature is fixed. In this sense, it is a canonical ensemble but an area ensemble as some ideas might suggest.

VI. CONCLUSIONS

In this paper, we have proposed a statistical fields theory underlying Perelman's seminal analogies between his geometric functionals and the thermodynamic functions. The theory is based on a $d = 4 - \epsilon$ quantum non-linear sigma model, interpreted as a quantum reference frame. When we quantize the theory at the Gaussian approximation, the wavefunction $\Psi(X)$ and hence the density matrix $u(X) = \Psi^*(X)\Psi(X)$ eq.(13) can be written down explicitly. Based on the density matrix, the Ricci flow of the frame fields (10) and the generalized Ricci-DeTurck flow (19) of the frame fields endowed with the density matrix is discussed. And further more, we find that the density matrix has profound statistical and geometric meanings, by using it, the spacetime (M^D, g) as the target space of NLSM is generalized to a density spacetime (M^D, g, u). The density matrix $u(X, \tau)$, satisfying a conjugate heat equation (23), not only describes a (coarse-grained) probability density of finding frame fields in a local volume, but also describes a volume comparison between a local volume and the fiducial one.

For the non-isometric nature of the Ricci or Ricci-DeTurck flow, the classical diffeomorphism is broken down at the quantum level. By the functional integral quantization method, the change of the measure of the functional integral can be given by using a Shannon entropy N in terms of the density matrix $u(X, \tau)$. The induced trace anomaly and its relation to the anomalies in conventional gravity theories are also discussed. As the Shannon entropy flows monotonically to its maximal value N_* in a limit called Gradient Shrinking Ricci Soliton (GSRS), a relative density $\tilde{u}$ and relative Shannon entropy $\tilde{N} = N - N_*$ can be defined w.r.t. the flow limit. The relative Shannon entropy gives a statistical interpretation underlying Perelman's partition function (47). And the monotonicity of $\tilde{N}$ along the Ricci flow gives an analogous H-theorem (50) for the frame fields system. As a side effect, the meanings on the gravitational side of the theory is also discussed, in which a cosmological constant $-\lambda\nu(B^4_\infty) \approx 0.8\rho_c$ as a UV counter term of the anomaly must be introduced.

We find that a temporal static GSRS, M^3, as a 3-space slice of the 4-spacetime GSRS, $M^4 = M^3 \times \mathbb{R}$, is in a thermal equilibrium state, in which the temperature is proportional to the global τ parameter of M^3 (73) up to a 3-space energy density λ_3 with normalization $\int d^3x\lambda_3 = 1$. The temperature and λ_3 both depend on the choice of time $\mathbb{R}$. In the sense that M^3 is in thermal, its Ricci soliton equation eq.(91) or quantum (indistinguishable with thermal) fluctuation eq.(41), can be considered as a generalization of the Unruh's formula, relating the temperature to local acceleration or gravitation. Based on the statistical interpretation of the density matrix $u(X, \tau)$, we find that the thermodynamic partition function (75) at the Gaussian approximation is just a partition function of ideal gas of the frame fields. In this physical picture of canonical ensemble of frame fields gas, several thermodynamic functions, including the internal energy (78), the free energy (84), the thermodynamic entropy (85), and the ensemble density (83) etc. can be calculated explicitly agreeing with Perelman's formulae, which gives an underlying statistical foundation to Perelman's analogous functionals.

We find that the statistical fields theory of quantum reference frame can be used to give a possible underlying microscopic origin of the spacetime thermodynamics. The standard results of the thermodynamics of the Schwarzschild black hole, including the Hawking temperature, energy and Bekenstein-Hawking entropy can be successfully reproduced in the framework. And we find that when the fluctuation internal energy of the metric is taken into account in the total energy, the energy of the black hole has a lower bound of order of the Planck energy, which avoid the quantum black hole evaporating into nothing. The internal energy or related temperature of the spacetime frame fields is an additional source of gravity, although macroscopically it is very small, at the quantum level its contribution is necessary for a thermal equilibrium 3-space just right being a GSRS, otherwise, the Equivalence Principle would breakdown. In this paper, the extended quantum Equivalence Principle plays a fundamental role as a bridge from the quantum reference frame theory (as a statistical fields or quantum fields theory on the base/lab spacetime) to the quantum gravity.

To sum up, the paper can be seen as an attempt to discuss the deep relations between these three fundamental themes: the diffeomorphism anomaly, gravity and the spacetime thermodynamics, based on the statistical fields

theory of quantum spacetime reference frame and the quantum Equivalence Principle. In the spirit of classical general relativity, if we trust the Equivalence Principle, one can not in principle figure out whether one is in an absolute accelerating frame or in an absolute gravitational background, which leads to a general covariance principle or diffeomorphism invariance of the gravitational theory. However, at the quantum level, the issue is a little subtle. If an observer in an accelerating frame sees the Unruh effect, i.e. thermal particles are creating in the "vacuum", which seems leading to the unitarily inequivalence between the vacuums of, for instance, an inertial frame and an accelerating frame, and hence the diffeomorphism invariance is seen breakdown discussed as the anomaly in the paper. The treatment of the anomaly in the paper is that, the anomaly is only canceled in an observer's lab up to UV scale, where the frame can be considered classical, rigid and cold, while at general scale the anomaly is not completely canceled. Whether one can figure out that he/she is in an absolute accelerating frame by detecting the anomaly (Shannon $\tilde{N}$ term) at general scale (e.g. by thermodynamic experiments detecting the vacuum thermal particle creation and hence find the non-unitarity)? We argue that if the answer is still "NO!" in the spirit of the general relativity, the anomaly term coming from a quantum general coordinates transformation must be also equivalently interpreted as the effects of spacetime thermodynamics and gravity. Because the 2nd order moment fluctuation of the quantum coordinates or a non-trivial manifolds density u, which gives rise to the diffemorphism anomaly, also contributes to other 2nd order quantities (series coefficients at second spacetime derivative) such as (i) the acceleration (second time derivative of coordinates, e.g. leading to uniform acceleration expansion or other acceleration discrepancies in the universe [1]), (ii) the gravity or curvature (second spacetime derivative of metric, e.g. see (9) and (18)) and (iii) the thermal broadening (second spatial derivative of the manifolds density or the ensemble density, e.g. see (41) and (73)) at the same (2nd) order. In this sense, the validity of the classical Equivalence Principle would be generalized to the quantum level to incorporate the effects of the quantum fluctuation of the spacetime coordinates or frame fields, so that, one in principle still can not figure out and distinguish whether he/she is in an accelerating frame, or in a gravitational field or in a thermal spacetime (as a new dimension of the Equivalence Principle), these three things have no absolute physical meanings and are indistinguishable any more in the framework. The classical Equivalence Principle asserts the equivalence of the first two things at the first order (mean level), the quantum Equivalence Principle asserts the equivalence of the three things even at the second order (variance level), even higher order.

Data availability statement

All data that support the findings of this study are included within the article.

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